

# A REPRESENTATION OF JOINT MOMENTS OF CUE CHARACTERISTIC POLYNOMIALS IN TERMS OF PAINLEVÉ FUNCTIONS

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**ABSTRACT.** We establish a representation of the joint moments of the characteristic polynomial of a CUE random matrix and its derivative in terms of a solution of the  $\sigma$ -Painlevé V equation. The derivation involves the analysis of a formula for the joint moments in terms of a determinant of generalised Laguerre polynomials using the Riemann-Hilbert method. We use this connection with the  $\sigma$ -Painlevé V equation to derive explicit formulae for the joint moments and to show that in the large-matrix limit the joint moments are related to a solution of the  $\sigma$ -Painlevé III' equation. Using the conformal block expansion of the  $\tau$ -functions associated with the  $\sigma$ -Painlevé V and the  $\sigma$ -Painlevé III' equations leads to general conjectures for the joint moments.

## 1. INTRODUCTION

Let  $U \in \mathbf{U}(N)$  be taken from the Circular Unitary Ensemble (CUE) of random matrices. Consider its characteristic polynomial,

$$(1-1) \quad Z_U(\theta) := \prod_{n=1}^N (1 - e^{i(\theta_n - \theta)}),$$

where  $e^{i\theta_1}, \dots, e^{i\theta_N}$  are the eigenvalues of  $U$  and  $\theta_i \in [0, 2\pi)$ . Put

$$(1-2) \quad V_U(\theta) := \exp \left( iN \frac{\theta + \pi}{2} - i \sum_{n=1}^N \frac{\theta_n}{2} \right) Z_U(\theta),$$

so that  $V_U(\theta)$  is real-valued for  $\theta \in [0, 2\pi)$ . The objects of our study are the joint moments of the function  $V_U(\theta)$  and its derivative,

$$(1-3) \quad F_N(h, k) := \int_{\mathbf{U}(N)} |V_U(0)|^{2k-2h} |V'_U(0)|^{2h} d\mu^{\text{Haar}},$$

where it is assumed that

$$h > -\frac{1}{2} \quad \text{and} \quad k > h - \frac{1}{2}.$$

Here  $d\mu^{\text{Haar}}$  is Haar measure, the unique probability measure over the  $N \times N$  unitary matrices that is invariant under the action of the unitary group [11].

These joint moments have been the focus of a number of previous studies: when  $h = 0$  they can be computed in several different ways – see, for example [32, 4]; similarly, when  $h = k$  they can be computed using standard techniques [10]; the general mixed moments for  $h, k \in \mathbb{N}$  have been analysed by combining these approaches [21, 22, 12, 13, 46, 44]. The results obtained suggest that in general  $F_N(h, k)$  grows like  $N^{k^2+2h}$  as  $N \rightarrow \infty$  and it is a key problem to prove this and then to evaluate the limit

$$(1-4) \quad F(h, k) := \lim_{N \rightarrow \infty} \frac{1}{N^{k^2+2h}} F_N(h, k).$$

For  $h, k \in \mathbb{N}$ ,  $k > h - 1/2$ , an expression for  $F_N(h, k)$  was obtained in [12] in terms of certain sums over partitions. A similar answer was also found in the case  $h = (2m - 1)/2$ ,  $m \in \mathbb{N}$ ,  $k \in \mathbb{N}$ ,

$k > h - 1/2$  in [46] (see [46] also for a survey on other related results). However, these formulae do not allow for easy computation beyond the first few values of  $k$  and  $h$ , in part because they are not recursive; also, they do not lead straightforwardly to formulae that extend to non-integer or non-half-integer values of  $k$  and  $h$ . The goal remains to analyse the large- $N$  limit of  $F_N(h, k)$  and evaluate the quantity  $F(h, k)$  for arbitrary real  $k$  and  $h$ .

One motivation for studying the joint moments is as follows. In 1973, Montgomery [38] conjectured that, assuming the Riemann hypothesis, the distances between appropriately normalised pairs of zeros of the Riemann zeta function follow a certain distribution previously shown by Dyson [15] to describe spacings between pairs of eigenvalues of unitary random matrices. Further evidence of a connection between number theory and random matrix theory was given when Keating and Snaith [32, 33] used results for the characteristic polynomial of a random unitary matrix to formulate conjectures about moments of the zeta function that are supported by number-theoretic and numerical results (see, for instance, the review articles [30, 45, 34, 31]). Since then, a number of more general results on the joint moments of the characteristic polynomial and its derivative have been proven and used to formulate conjectures about the joint moments of  $L$ -functions and their derivatives [7, 8, 9, 10, 12, 23, 24, 46].

Our objective is to connect the joint moments of characteristic polynomials with the theory of Painlevé equations (see also [3] for a recent result connecting Painlevé functions with random matrix spacing distributions related to zeta zeros in a different direction). Solutions to the Painlevé equations play an important role in many aspects of random matrix theory (see, for instance, [19, 26]) and are amenable to asymptotic analysis [14, 16].

The results presented here relate  $F_N(h, k)$  to a particular solution of the  $\sigma$ -Painlevé V equation and the limiting function (1-4) to a particular solution of the  $\sigma$ -Painlevé III' equation. We see them as the first step in a longer-term project to use asymptotic analysis of Painlevé functions and related objects from the theory of integrable systems to obtain stronger asymptotic results on joint moments of characteristic polynomials.

**1.1. Results.** Let  $L_n^{(\alpha)}(s)$  be the generalised Laguerre polynomial

$$(1-5) \quad L_n^{(\alpha)}(s) := \frac{e^s}{s^{\alpha} n!} \frac{d^n}{ds^n} (s^{\alpha+n} e^{-s}) = \sum_{j=0}^n \frac{\Gamma(n + \alpha + 1)}{\Gamma(j + \alpha + 1)(n - j)!} \frac{(-s)^j}{j!}$$

and define

$$(1-6) \quad K_n(\epsilon, y) := \frac{(-1)^n}{\pi} \frac{\partial^n}{\partial \epsilon^n} \left( \frac{\epsilon}{\epsilon^2 + y^2} \right).$$

As we will explain in §2,  $F_N(h, k)$  is related to the generalised Laguerre polynomials by

$$(1-7) \quad F_N(h, k) = \lim_{\epsilon \rightarrow 0} (-1)^{\frac{k(k-1)}{2}} 2^{-2h} \int_{-\infty}^{\infty} K_{2h}(\epsilon, y) e^{-N|y|} \det \left[ L_{N+k-1-(i+j)}^{(2k-1)}(-2|y|) \right]_{i,j=0,\dots,k-1} dy,$$

with  $N > k - 1$ . Our main theorem relates the Laguerre determinant to a specific solution of the equation

$$(1-8) \quad \left( x \frac{d^2 \sigma}{dx^2} \right)^2 = \left( \sigma - x \frac{d\sigma}{dx} + 2 \left( \frac{d\sigma}{dx} \right)^2 - 2N \frac{d\sigma}{dx} \right)^2 - 4 \frac{d\sigma}{dx} \left( -N + \frac{d\sigma}{dx} \right) \left( -k - N + \frac{d\sigma}{dx} \right) \left( k + \frac{d\sigma}{dx} \right).$$

Equation (1-8) is a special case of the  $\sigma$ -Painlevé V equation with three parameters given in (3-21) (see Okamoto [40, 41, 42] and Jimbo and Miwa [27]). For further examples of solutions of Painlevé equations expressed as Wronskian determinants of generalised Laguerre polynomials or confluent hypergeometric functions see [17, 29, 36, 39].

**Theorem 1.** *We have the representation*

$$(1-9) \quad \det \left[ L_{N+k-1-(i+j)}^{(2k-1)}(-2|y|) \right]_{i,j=0,\dots,k-1} = \frac{e^{-2k|y|}}{(2\pi i)^k} H_k[w_0],$$

where  $H_k[w_0] = H_n[w_0]|_{n=k}$ , and  $H_n[w_0]$  is the Hankel determinant

$$(1-10) \quad H_n[w_0] := \det \left[ \int_C w_0(s) s^{i+j} ds \right]_{i,j=0,\dots,n-1}$$

with the weight

$$(1-11) \quad w_0(s) := \frac{e^{\frac{x}{1-s}}}{(1-s)^{2k} s^{N+k}}, \quad x = 2|y|.$$

Here  $C$  is a small (radius less than 1) positively oriented circle around zero. Furthermore,

$$(1-12) \quad \frac{d}{dx} \log H_k = \frac{\sigma(x) + kx + Nk}{x},$$

where  $\sigma(x)$  is a solution of the  $\sigma$ -Painlevé V equation (1-8) with asymptotics

$$(1-13) \quad \sigma(x) = -Nk + \frac{N}{2}x + \mathcal{O}(x^2), \quad x \rightarrow 0.$$

Theorem 1 is proven in §3.

The solution of the  $\sigma$ -Painlevé V equation considered here is a rational solution. Rational solutions of Painlevé equations have been obtained in [5, 6, 37] in terms of Wronskian determinants of confluent hypergeometric functions which include also the case of generalised Laguerre polynomials. The relation between the Wronskian determinant in [6] and our Hankel determinant formula (1-10) is not immediate. For this reason we here provide an alternate proof that is quite straightforward and algorithmic to show that such determinant is a particular solution of the Painlevé V equation. Furthermore, we show that  $F_N(h, k)$  can be evaluated recursively from equation (1-8) for integer values of  $h$ , giving formulae that extend to all  $k$ . Using the conformal block expansion of the  $\tau$  function of the Painlevé V equation introduced by Lisovsky, Nagoya, and Roussillon [35], we give a combinatorial expression of the coefficients  $F_N(h, k)$  (see §6).

Our second main result concerns the evaluation of the function  $F(h, k)$  in (1-4) for  $h, k \in \mathbb{N}$ ,  $k > h - 1/2$ . Let  $\xi(t)$  be the particular solution of the equation

$$(1-14) \quad \left( t \frac{d^2 \xi}{dt^2} \right)^2 = -4t \left( \frac{d\xi}{dt} \right)^3 + (4k^2 + 4\xi) \left( \frac{d\xi}{dt} \right)^2 + t \frac{d\xi}{dt} - \xi,$$

with initial conditions

$$(1-15) \quad \xi(0) = 0, \quad \xi'(0) = 0$$

where prime denotes derivative with respect to  $t$ . Equation (1-14) is a special case of the  $\sigma$ -Painlevé III' equation with two parameters (cf. (4-34)). For background on the  $\sigma$ -Painlevé III' equation see Okamoto [40, 41, 43] and Jimbo and Miwa [27]. We show in Theorem 2, which we state in §4, that

$$(1-16) \quad F(h, k) = (-1)^h \frac{G(k+1)^2}{G(2k+1)} \frac{d^{2h}}{dt^{2h}} \left[ \exp \int_0^t \left( \frac{\xi(s)}{s} ds \right) \right] \Big|_{t=0},$$

where  $G$  is the Barnes function (see Appendix C). Furthermore, introducing the  $\tau$ -function of the Painlevé III equation defined as  $t \frac{d}{dt} \log \tau_{III}(t) = \xi(t)$ , we have

$$F(h, k) = (-1)^h \frac{d^{2h}}{dt^{2h}} \tau_{III}(t) \Big|_{t=0}.$$

Using the conformal block expansion of  $\tau_{III}(t)$  near  $t = 0$  we arrive at the conjectural expression (1-17)

$$F(h, k) = (-1)^h \frac{G(k+1)^2}{G(2k+1)} (2h)! \sum_{\substack{\lambda \in \mathbb{Y} \\ |\lambda| = 2h, \lambda_1 \leq k}} \prod_{(i,j) \in \lambda} \frac{(2k+i-j)(k+i-j)}{h_\lambda^2(i,j) \left( \lambda'_j - i - j + 1 + 2k \right)^2}, \quad k > h - \frac{1}{2},$$

where the sum is taken over all Young diagrams with  $2h$  boxes and first row with at most  $k$  boxes,  $h_\lambda(i, j)$  is the hook length of the box  $(i, j)$  associated to the diagram  $\lambda$ , and  $\lambda'_j$  is the number of boxes in the  $j$  column of the transpose diagram  $\lambda'$  (see §6). The values of the above coefficients for small values of  $h$  are consistent with the results obtained by Dehaye in [12].

The connection between moments of characteristic polynomials of the unitary group and the Painlevé III' equation has already appeared in the literature. In particular, defining the characteristic polynomial of the unitary matrix  $A$  to be

$$(1-18) \quad \Lambda_A(s) := \det[I - sA^*] = \prod_{n=1}^N (1 - se^{-i\theta_n}),$$

Conrey, Rubinstein, and Snaith proved the following.

**Previous result** (Conrey, Rubinstein, and Snaith [10]). *For fixed  $k$  and  $N \rightarrow \infty$ ,*

$$(1-19) \quad \int_{U(N)} |\Lambda'_A(1)|^{2k} dA_N = b_k N^{k^2+2k} + \mathcal{O}\left(N^{k^2+2k-1}\right),$$

where

$$(1-20) \quad b_k := (-1)^{k(k+1)/2} \sum_{h=0}^k \binom{k}{h} \left( \frac{d}{dt} \right)^{k+h} \left( e^{-t} t^{-k^2/2} \det_{k \times k} [I_{i+j-1}(2\sqrt{t})] \right) \Big|_{t=0}.$$

Here  $I_\nu(z)$  is the modified Bessel function of the first kind.

Forrester and Witte [18] then showed that

$$(1-21) \quad b_k = \frac{(-1)^k}{k!} \prod_{j=1}^k \frac{j!}{(j+k-1)!} \sum_{h=0}^k \binom{k}{h} \left( \frac{d}{dt} \right)^{k+h} \tau_k(t) \Big|_{t=0},$$

where  $\tau_k(t)$  is a  $\tau$  function of the Painlevé III' equation.

In work independent of ours, the method of [10] has recently been extended to the joint moments of  $\Lambda_A(s)$  in [1], yielding a result equivalent to our Theorem 2.

**1.2. Next steps.** As mentioned already, we see our result as providing a new starting point from which one can attempt to derive large- $N$  asymptotics and so seek to prove (1-4) and evaluate  $F(h, k)$  for general  $h$  and  $k$ . In §6 we present the conformal block expansion of the  $\tau$  function of the Painlevé V equation [35] and Painlevé III equation [20]. Such expansions could be potentially used to obtain formulae for  $F_N(h, k)$  and  $F(h, k)$  beyond integer values of  $k$ .

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## 2. INTEGRAL REPRESENTATIONS FOR $F_N(h, k)$

Our starting formulae are integral representations for  $F_N(h, k)$  obtained in [46] for integer  $k$  and integer or half-integer  $h$ . The first formula involves an  $(N + 1)$ -fold integral.

**Proposition 1** (Proposition 1 of [46]). Let  $n \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ , and define  $K_n(\epsilon, y)$  by (1-6). Then, if  $2h \in \mathbb{N}_0$  and  $k > h - \frac{1}{2}$ ,

$$(2-1) \quad F_N(h, k) = \lim_{\epsilon \rightarrow 0} \frac{2^{N^2+2kN-2h}}{(2\pi)^N N!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} K_{2h}(\epsilon, y) \prod_{j=1}^N \frac{e^{iyx_j}}{(1+x_j^2)^{N+k}} \Delta^2(\mathbf{x}) d\mathbf{x} dy,$$

where  $\Delta^2(\mathbf{x}) \equiv \Delta^2(x_1, \dots, x_N)$  is the Vandermonde determinant,

$$\Delta^2(\mathbf{x}) := \prod_{1 \leq j < k \leq N} (x_k - x_j).$$

We emphasise that formula (2-1) holds for any real  $k > h - \frac{1}{2}$ . As is also shown in [46], in the case of integer  $k$  the  $\mathbf{x}$ -integral in the right-hand side of (2-1) can be evaluated in terms of Laguerre polynomials. Put

$$H(k, y) := \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{j=1}^N \frac{e^{iyx_j}}{(1+x_j^2)^{N+k}} \Delta^2(\mathbf{x}) d\mathbf{x}.$$

**Proposition 2** (Proposition 4 of [46]). For  $k \in \mathbb{N}$  and  $y \in \mathbb{R}$ ,

$$(2-2) \quad H(k, y) = (-1)^{\frac{k(k-1)}{2}} \frac{(2\pi)^N N!}{2^{2kN+N^2}} e^{-N|y|} \det \left[ L_{N+k-1-(i+j)}^{(2k-1)}(-2|y|) \right]_{i,j=0,\dots,k-1}.$$

This in turn implies that for integer  $k > h - \frac{1}{2}$  the  $(N + 1)$ -fold integral representation (2-1) for  $F_N(h, k)$  can be transformed into the single integral formula (1-7). This equation is our starting formula. It was also the starting point of [46], which, for example, computed  $F_N(h, k)$  explicitly in the case  $h = \frac{1}{2}$  and  $k = 1$ , proving the limit in (1-4) exists and showing that

$$(2-3) \quad F\left(\frac{1}{2}, 1\right) = \frac{e^2 - 5}{4\pi}.$$

## 3. PAINLEVÉ V AND THE LAGUERRE DETERMINANT

In this section we first introduce the Painlevé V equation and its  $\tau$  function. The goal of the section is to show that the determinant (1-10) corresponds to a particular solution of the Painlevé V equation.

**3.1. The Painlevé V equation and its  $\tau$ -function.** In this section we summarise the main properties of the Painlevé V equation that can be found in [27]. The general Painlevé V equation for a complex function  $y = y(x)$  takes the form

$$(3-1) \quad \frac{d^2 y}{dx^2} = \left( \frac{1}{2y} + \frac{1}{y-1} \right) \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} + \frac{(y-1)^2}{x^2} \left( \alpha y + \frac{\beta}{y} \right) + \gamma \frac{y}{x} + \delta \frac{y(y+1)}{y-1}.$$

The coefficients  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  are complex constants. One can fix  $\delta = -\frac{1}{2}$  because the general Painlevé V equation with  $\delta \neq 0$  can be reduced to the case with  $\delta = -\frac{1}{2}$  by the mapping  $x \mapsto \sqrt{-2\delta}x$ .

The equation (3-1) has a Lax pair, namely it can be written as the compatibility condition of two linear systems of ODEs for the  $2 \times 2$  matrix function  $\Phi(z, x)$ ,  $z, x \in \mathbb{C}$ , that satisfies the equations

$$(3-2) \quad \frac{\partial \Phi}{\partial z} = \left( \frac{x}{2} \sigma_3 + \frac{A_0}{z} + \frac{A_1}{z-1} \right) \Phi(z, x), \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$(3-3) \quad \frac{\partial \Phi}{\partial x} = \left( \frac{z}{2} \sigma_3 + \frac{B_0}{x} \right) \Phi(z, x),$$

with

$$(3-4) \quad A_0 := \begin{pmatrix} w + \frac{\theta_0}{2} & -u(w + \theta_0) \\ u^{-1}w & -w - \frac{\theta_0}{2} \end{pmatrix},$$

$$(3-5) \quad A_1 := \begin{pmatrix} -w - \frac{\theta_0 + \theta_\infty}{2} & uy \left( w + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) \\ -(uy)^{-1} \left( w + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) & w + \frac{\theta_0 + \theta_\infty}{2} \end{pmatrix},$$

and

$$(3-6) \quad B_0 := \begin{pmatrix} 0 & -u(w + \theta_0) + uy \left( w + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) \\ u^{-1}w - (uy)^{-1} \left( w + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) & 0 \end{pmatrix} \frac{1}{x},$$

where  $\theta_j$ ,  $j = 0, 1, \infty$  are constant parameters and  $u \equiv u(x)$ ,  $w \equiv w(x)$ , and  $y \equiv y(x)$ . As shown in [27], the compatibility condition of equations (3-2) and (3-3), namely

$$(3-7) \quad \frac{\partial}{\partial x} \frac{\partial \Phi}{\partial z} = \frac{\partial}{\partial z} \frac{\partial \Phi}{\partial x},$$

implies that the functions  $w, y, u$  satisfy the following  $3 \times 3$  system of first-order ordinary differential equations (cf. [27, (C.40)]):

$$(3-8) \quad x \frac{dy}{dx} = xy - 2w(y-1)^2 - (y-1) \left( \frac{\theta_0 - \theta_1 + \theta_\infty}{2} y - \frac{3\theta_0 + \theta_1 + \theta_\infty}{2} \right),$$

$$(3-9) \quad x \frac{dw}{dx} = yw \left( w + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) - \frac{1}{y} (w + \theta_0) \left( w + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right),$$

$$(3-10) \quad x \frac{d}{dx} \log u = -2w - \theta_0 + y \left( w + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) + \frac{1}{y} \left( w + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right).$$

Equation (3-10) just gives  $u$  in terms of  $y$  and  $w$ , while the system of two equations (3-8)–(3-9) is, in fact, equivalent to the fifth Painlevé equation (3-1) for the function  $y(x)$ , where

$$(3-11) \quad \alpha = \frac{1}{2} \left( \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right)^2, \quad \beta = -\frac{1}{2} \left( \frac{\theta_0 - \theta_1 - \theta_\infty}{2} \right)^2, \quad \gamma = 1 - \theta_0 - \theta_1, \quad \delta = -\frac{1}{2}.$$

We observe that the matrices  $A_0$ ,  $A_1$ , and  $B_0$  in (3-4), (3-5), and (3-6) are traceless and the eigenvalues of  $A_0$  and  $A_1$  are constants:

$$(3-12) \quad \text{Spect } A_0 = \left\{ \pm \frac{\theta_0}{2} \right\}$$

and

$$(3-13) \quad \text{Spect } A_1 = \left\{ \pm \frac{\theta_1}{2} \right\}.$$

It follows that the solution of equation (3-2) in the neighbourhood of the regular singular points  $z = 0$  and  $z = 1$  takes the form

$$(3-14) \quad \Phi(z) = \hat{\Phi}_0(z) z^{\frac{\theta_0}{2} \sigma_3}, \quad z \sim 0,$$

$$(3-15) \quad \Phi(z) = \widehat{\Phi}_1(z)(z-1)^{\frac{\theta_1}{2}\sigma_3}, \quad z \sim 1,$$

where  $\widehat{\Phi}_0(z)$  and  $\widehat{\Phi}_1(z)$  are holomorphic and invertible in neighbourhoods of the respective points. Regarding the behaviour of the solution of (3-2) near the irregular singular point  $z = \infty$  of Poincaré rank 1, one has to consider the diagonal part,  $\text{diag}(A_0 + A_1) = -\frac{\theta_\infty}{2}\sigma_3$ . It follows that the formal solution of  $\Phi(z)$  in the neighbourhood of  $z = \infty$  takes the form [27]

$$(3-16) \quad \Phi(z) = \widehat{\Phi}_\infty(z)e^{\frac{xz}{2}\sigma_3}z^{\frac{\theta_\infty}{2}\sigma_3}, \quad z \sim \infty,$$

where  $\widehat{\Phi}_\infty(z)$  has the formal asymptotic expansion at  $z = \infty$

$$(3-17) \quad \widehat{\Phi}_\infty(z) = I + \frac{\phi_1}{z} + \dots, \quad z \sim \infty,$$

where  $\phi_1$  is a matrix independent of  $z$ . We define the Hamiltonian  $\mathcal{H}$  as

$$(3-18) \quad \mathcal{H} := -(\phi_1)_{11},$$

where  $(\phi_1)_{11}$  denotes the 11 entry of the matrix  $\phi_1$ . In fact, one has

$$(3-19) \quad \mathcal{H} := -\frac{1}{x} \left( w - \frac{1}{y} \left( w + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) \right) \left( w + \theta_0 - y \left( w + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) \right) - w - \frac{\theta_0 + \theta_\infty}{2}.$$

Up to linear terms in  $x$ , the function  $x\mathcal{H}$  satisfies a second-order ODE which is called the  $\sigma$ -form of the Painlevé V equation. More precisely, defining the function

$$(3-20) \quad \sigma := x\mathcal{H} + \frac{1}{2}(\theta_0 + \theta_\infty)x + \frac{1}{4}(\theta_0 + \theta_\infty)^2 - \frac{\theta_1^2}{4},$$

then

$$(3-21) \quad \left( x \frac{d^2\sigma}{dx^2} \right)^2 = \left[ \sigma - x \frac{d\sigma}{dx} + 2 \left( \frac{d\sigma}{dx} \right)^2 - (2\theta_0 + \theta_\infty) \frac{d\sigma}{dx} \right]^2 - 4 \frac{d\sigma}{dx} \left( \frac{d\sigma}{dx} - \theta_0 \right) \left( \frac{d\sigma}{dx} - \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) \left( \frac{d\sigma}{dx} - \theta_0 - \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right).$$

Finally, the  $\tau$ -function is defined (cf. [27, 28]) in terms of the Hamiltonian  $\mathcal{H}$  by

$$(3-22) \quad \mathcal{H} =: \frac{d}{dx} \log \tau,$$

so that, by (3-20),

$$\sigma = x \frac{d}{dx} \log \left( x^{\frac{1}{4}(\theta_0 + \theta_\infty)^2 - \frac{1}{4}\theta_1^2} e^{\frac{1}{2}(\theta_0 + \theta_\infty)x} \tau(x) \right).$$

In the next subsection we show that the Hankel determinant (1-10) is a tau-function of the Painlevé V equation. We will proceed as follows:

1. We formulate a Riemann-Hilbert problem for the generalised Laguerre polynomials (1-5) and we derive a system of ODEs related to this Riemann-Hilbert problem;
2. We introduce a series of rational and gauge transformations to reduce the system of ODEs obtained in 1. to the Lax pair (3-2)–(3-3);
3. Finally, we identify the Hankel determinant (1-10) with the  $\tau$ -function of the Painlevé V equation via the relation (3-20).



**3.2. Laguerre determinant.** We will analyse the principal ingredient of the starting formula (1-7), i.e. the *Laguerre determinant*

$$(3-23) \quad \det \left[ L_{N+k-1-(i+j)}^{(2k-1)}(-2|y|) \right]_{i,j=0,\dots,k-1}.$$

Recall the classical formula

$$(3-24) \quad L_n^{(\alpha)}(x) = \int_C \frac{e^{-xt/(1-t)}}{(1-t)^{\alpha+1} t^{n+1}} dt$$

for the generalised Laguerre polynomials, where  $C$  is a closed contour around 0 (for instance, the positively oriented circle around 0 with radius  $\frac{1}{2}$ ). In our case, we have

$$(3-25) \quad \begin{aligned} L_{N+k-1-(i+j)}^{(2k-1)}(x) &= \frac{1}{2\pi i} \int_C \frac{e^{-\frac{xt}{1-t}}}{(1-t)^{2k} t^{N+k-(i+j)}} dt \\ &= \frac{1}{2\pi i} \int_C \frac{e^{-\frac{xt}{1-t}}}{(1-t)^{2k} t^{N+k}} t^{i+j} dt = \frac{1}{2\pi i} e^x \int_C \frac{e^{-\frac{x}{1-t}}}{(1-t)^{2k} t^{N+k}} t^{i+j} dt. \end{aligned}$$

Hence

$$(3-26) \quad \det \left[ L_{N+k-1-(i+j)}^{(2k-1)}(-2|y|) \right]_{i,j=0,\dots,k-1} = \frac{e^{-2k|y|}}{(2\pi i)^k} H_k[w_0],$$

where

$$(3-27) \quad H_n[w_0] := \det \left[ \int_C t^{i+j} w_0(t) dt \right]_{i,j=0,\dots,n-1}$$

is the Hankel determinant with the weight

$$(3-28) \quad w_0(t) = \frac{e^{2|y|/(1-t)}}{(1-t)^{2k} t^{N+k}}.$$

Following the general Riemann-Hilbert scheme in the theory of Hankel determinants (see e.g. [2, 25]), we consider the system of monic orthogonal polynomials with weight  $w_0(t)$  on  $C$ ,

$$P_n(t) = t^n + \dots, \quad \int_C P_n(t) t^m w_0(t) dt = h_n \delta_{nm}, \quad m = 0, \dots, n,$$

and define the  $2 \times 2$  matrix valued function

$$(3-29) \quad Y(t) := \begin{pmatrix} P_n(t) & \frac{1}{2\pi i} \int_C \frac{P_n(t') w_0(t') dt'}{t' - t} \\ -\frac{2\pi i}{h_{n-1}} P_{n-1}(t) & -\frac{1}{h_{n-1}} \int_C \frac{P_{n-1}(t') w_0(t') dt'}{t' - t} \end{pmatrix}.$$

The defining property of the function  $Y(t)$  is that it is the unique solution of the following matrix Riemann-Hilbert problem:

$$(3-30) \quad Y(t) \in \mathcal{H}(\mathbb{C} \setminus C),$$

$$(3-31) \quad Y_+(t) = Y_-(t) \begin{pmatrix} 1 & \frac{e^{-\frac{x}{1-t}}}{(1-t)^{2k} t^{N+k}} \\ 0 & 1 \end{pmatrix}, \quad t \in C, \quad x = 2|y|,$$

$$(3-32) \quad Y(t) = \left( I + \mathcal{O}\left(\frac{1}{t}\right) \right) t^{n\sigma_3}, \quad t \rightarrow \infty,$$



where  $\mathcal{H}(\mathbb{C} \setminus C)$  stands for the holomorphic  $2 \times 2$  matrix-valued functions in  $\mathbb{C} \setminus C$ . The Hankel determinant  $H_n[w_0]$  is related to the function  $Y(t)$  via the equations

$$(3-33) \quad \frac{H_{n+1}}{H_n} = h_n, \quad h_n = -2\pi i(m_1)_{12}, \quad \frac{1}{h_{n-1}} = -\frac{1}{2\pi i}(m_1)_{21},$$

where the matrix  $m_1$  is the first coefficient in the expansion (3-32), i.e.,

$$(3-34) \quad Y(t) \sim \left(I + \frac{m_1}{t} + \dots\right) t^{n\sigma_3}, \quad t \rightarrow \infty.$$

It also should be noticed that

$$(3-35) \quad \det Y(t) \equiv 1.$$

Let us change the variable  $t$  to  $z$ :  $z = \frac{1}{1-t}$ ,  $t = \frac{z-1}{z}$  so that the circle  $C$  maps to the new circle

$$(3-36) \quad \Gamma := \left\{ z : \left| z - \frac{4}{3} \right| = \frac{2}{3} \right\},$$

oriented counterclockwise. Put

$$(3-37) \quad X(z) := Y(t(z)) \equiv Y\left(\frac{z-1}{z}\right).$$

Then, in terms of the function  $X(z)$ , the Riemann-Hilbert problem (3-30)–(3-32) reads as follows:

$$(3-38) \quad X(z) \in \mathcal{H}(\mathbb{C}P^1 \setminus (\Gamma \cup \{0\})),$$

$$(3-39) \quad X_+(z) = X_-(z) \begin{pmatrix} 1 & e^{xz} \frac{z^{3k+N}}{(z-1)^{N+k}} \\ 0 & 1 \end{pmatrix}, \quad z \in \Gamma,$$

$$(3-40) \quad X(z) \sim (I + m_1^X z + \dots) (-z)^{-n\sigma_3}, \quad z \rightarrow 0,$$

where

$$(3-41) \quad m_1^X := -m_1 - n\sigma_3,$$

and also

$$(3-42) \quad \det X(t) \equiv 1.$$

From the point of view of the modern theory of isomonodromic deformations, the Riemann-Hilbert problem (3-37)–(3-40) indicates that we are dealing with the fifth Painlevé equation. In what follows we present the detailed derivation of this fact.

Define the function

$$(3-43) \quad \Psi(z) := X(z) \begin{pmatrix} e^{\frac{xz}{2}} & 0 \\ 0 & e^{-\frac{xz}{2}} \frac{(z-1)^{k+N}}{z^{N+3k}} \end{pmatrix}.$$

Then,

$$(3-44) \quad \Psi(z), \Psi^{-1}(z) \in \mathcal{H}(\mathbb{C} \setminus (\Gamma \cup \{0\} \cup \{1\}))$$

and the function  $\Psi(z)$  has a constant jump across the circle  $\Gamma$ ,

$$(3-45) \quad \Psi_+(z) = \Psi_-(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad z \in \Gamma.$$

Moreover, in neighbourhoods of the points  $\infty$ ,  $0$ , and  $1$ , the function  $\Psi(z)$  exhibits the following behaviour:

$$(3-46) \quad \Psi(z) = \widehat{\Psi}_\infty(z) \begin{pmatrix} e^{\frac{xz}{2}} & 0 \\ 0 & e^{-\frac{xz}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^{-2k} \end{pmatrix}, \quad z \sim \infty,$$

$$(3-47) \quad \Psi(z) = \widehat{\Psi}_0(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n-3k-N} \end{pmatrix}, \quad z \sim 0,$$

$$(3-48) \quad \Psi(z) = \widehat{\Psi}_1(z) \begin{pmatrix} 1 & 0 \\ 0 & (z-1)^{N+k} \end{pmatrix}, \quad z \sim 1,$$

where  $\widehat{\Psi}_\infty(z)$ ,  $\widehat{\Psi}_0(z)$ , and  $\widehat{\Psi}_1(z)$  are holomorphic and invertible in the neighbourhoods of the respective points. Also,

$$(3-49) \quad \widehat{\Psi}_\infty(\infty) = X(\infty) = Y(1),$$

$$(3-50) \quad \widehat{\Psi}_0(0) = (-1)^{-n\sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{k+N} \end{pmatrix} = \begin{pmatrix} (-1)^{-n} & 0 \\ 0 & (-1)^{k+N+n} \end{pmatrix},$$

$$(3-51) \quad \widehat{\Psi}_1(1) = X(1)e^{\frac{x}{2}\sigma_3} = Y(0)e^{\frac{x}{2}\sigma_3}.$$

We should mention that the invertibility of the functional factors  $\widehat{\Psi}_j(z)$ ,  $j = 0, 1, \infty$  follows from the equation

$$(3-52) \quad \det \Psi(z) = \frac{(z-1)^{k+N}}{z^{N+3k}},$$

which in turn is a consequence of (3-42).

By standard arguments, the properties (3-44)–(3-48) imply that the function  $\Psi(z) \equiv \Psi(z, x)$  satisfies linear differential equations with respect to  $z$  and  $x$  of the form

$$(3-53) \quad \frac{\partial \Psi}{\partial z} = \left( x\widehat{A}_\infty + \frac{\widehat{A}_0}{z} + \frac{\widehat{A}_1}{z-1} \right) \Psi,$$

$$(3-54) \quad \frac{\partial \Psi}{\partial x} = (z\widehat{B}_\infty + \widehat{B}_0) \Psi,$$

where

$$(3-55) \quad \widehat{B}_0 = 0,$$

$$(3-56) \quad \widehat{A}_\infty = \widehat{B}_\infty = \frac{1}{2}X(\infty)\sigma_3X^{-1}(\infty) = \frac{1}{2}Y(1)\sigma_3Y^{-1}(1),$$

$$(3-57) \quad \widehat{A}_0 = \begin{pmatrix} -n & 0 \\ 0 & n-3k-N \end{pmatrix},$$

$$(3-58) \quad \widehat{A}_1 = X(1) \begin{pmatrix} 0 & 0 \\ 0 & N+k \end{pmatrix} X^{-1}(1) = Y(0) \begin{pmatrix} 0 & 0 \\ 0 & N+k \end{pmatrix} Y^{-1}(0).$$

Indeed, since the jump matrix in (3-45) is constant, the logarithmic derivatives  $\frac{\partial \Psi}{\partial z}\Psi^{-1}(z)$  and  $\frac{\partial \Psi}{\partial x}\Psi^{-1}(z)$  do not have jumps across  $\Gamma$  and hence are analytic in  $\mathbb{C} \setminus (\{0\} \cup \{1\})$ . In addition, formulae (3-46)–(3-48) tell us that  $\frac{\partial \Psi}{\partial z}\Psi^{-1}(z)$  has simple poles at  $z = 0$ ,  $z = 1$  and is holomorphic at  $z = \infty$  while  $\frac{\partial \Psi}{\partial x}\Psi^{-1}(z)$  is holomorphic at  $z = 0$ ,  $z = 1$  and has a simple pole at  $z = \infty$ . These arguments yield equations (3-53) and (3-54). Moreover, as  $z \rightarrow \infty$ ,

$$(3-59) \quad \frac{\partial \Psi}{\partial z}\Psi^{-1}(z) = \frac{x}{2}\widehat{\Psi}_\infty(\infty)\sigma_3\widehat{\Psi}_\infty^{-1}(\infty) + \mathcal{O}\left(\frac{1}{z}\right)$$

and

$$(3-60) \quad \frac{\partial \Psi}{\partial x} \Psi^{-1}(z) = \frac{z}{2} \widehat{\Psi}_{\infty}(\infty) \sigma_3 \widehat{\Psi}_{\infty}^{-1}(\infty) + \mathcal{O}(1)$$

which, together with (3-49), imply (3-56). Similarly, as  $z \rightarrow 0$ ,

$$\frac{\partial \Psi}{\partial z} \Psi^{-1}(z) = \frac{1}{z} \widehat{\Psi}_0(0) \begin{pmatrix} -n & 0 \\ 0 & n - 3k - N \end{pmatrix} \widehat{\Psi}_0^{-1}(0) + \mathcal{O}(1)$$

and, as  $z \rightarrow 1$ ,

$$\frac{\partial \Psi}{\partial z} \Psi^{-1}(z) = \frac{1}{z-1} \widehat{\Psi}_1(1) \begin{pmatrix} 0 & 0 \\ 0 & N+k \end{pmatrix} \widehat{\Psi}_1^{-1}(1) + \mathcal{O}(1)$$

which, together with (3-50) and (3-51), imply (3-57) and (3-58), respectively. Finally, as  $z \rightarrow 0$ , we have that

$$\frac{\partial \Psi}{\partial x} \Psi^{-1}(z) = \frac{\partial \widehat{\Psi}_0(z)}{\partial x} \widehat{\Psi}_0^{-1}(z) = \frac{\partial \widehat{\Psi}_0(0)}{\partial x} \widehat{\Psi}_0^{-1}(0) + \mathcal{O}(z) = \mathcal{O}(z),$$

which implies equation (3-55).

The matrix equation (3-53) is a  $2 \times 2$  system with rational coefficients having three singular points: two Fuchsian points at  $z = 0$  and  $z = 1$ , and one irregular singular point with Poincaré index 1 at  $z = \infty$ . The presence of the second matrix equation (3-54) shows that the  $x$ -dependence of the coefficients of system (3-53) is monodromy-preserving. In other words, we are dealing with the Painlevé-type isomonodromy deformation of (3-53). In fact, the pair of matrix equations (3-53)–(3-54) is almost the Lax pair (3-2)–(3-3) for the fifth Painlevé equation (3-1) given by Jimbo-Miwa [27]. To make it exactly the Jimbo-Miwa Painlevé V Lax pair a little extra work is needed. We observe that the matrices  $\widehat{A}_1$  and  $\widehat{A}_0$  are not traceless since

$$\text{Trace } \widehat{A}_0 = -3k - N, \quad \text{Trace } \widehat{A}_1 = N + k$$

and  $\Psi(z, x)$  is not normalised to the identity at  $z = \infty$ . We make the transformation

$$(3-61) \quad \Phi(z, x) := z^{\frac{3k+N}{2}} (z-1)^{-\frac{N+k}{2}} \widehat{\Psi}_{\infty}^{-1}(\infty) \Psi(z, x) = z^{\frac{3k+N}{2}} (z-1)^{-\frac{N+k}{2}} Y^{-1}(1) \Psi(z, x)$$

that brings the original system (3-53)–(3-54) to the normalised-at-infinity and traceless form (3-2)–(3-3), where

$$(3-62) \quad A_0 = Y^{-1}(1) \widehat{A}_0 Y(1) + \frac{3k+N}{2} I = Y^{-1}(1) \begin{pmatrix} -n + \frac{3k+N}{2} & 0 \\ 0 & n - \frac{3k+N}{2} \end{pmatrix} Y(1),$$

$$(3-63) \quad A_1 = Y^{-1}(1) \widehat{A}_1 Y(1) - \frac{k+N}{2} I = Y^{-1}(1) Y(0) \begin{pmatrix} -\frac{k+N}{2} & 0 \\ 0 & \frac{k+N}{2} \end{pmatrix} Y^{-1}(0) Y(1),$$

and

$$(3-64) \quad B_0 = -Y^{-1}(1) Y_x(1).$$

We also notice that the tracelessness of the coefficients of the matrix  $B_0$  follows from the identity  $\det Y(t) \equiv 1$ . To identify the matrices  $A_0$  and  $A_1$  in (3-62) and (3-63) with those defined in (3-4) and (3-5) we need some extra work. To this end we notice that the local equations (3-46)–(3-48) in terms of the new function  $\Phi(z)$  read

$$(3-65) \quad \Phi(z) = \widehat{\Phi}_{\infty}(z) e^{\frac{xz}{2} \sigma_3} z^{k\sigma_3}, \quad z \sim \infty,$$

$$(3-66) \quad \Phi(z) = \widehat{\Phi}_0(z) z^{(-n + \frac{3k+N}{2}) \sigma_3}, \quad z \sim 0,$$

$$(3-67) \quad \Phi(z) = \widehat{\Phi}_1(z)(z-1)^{-\frac{N+k}{2}\sigma_3}, \quad z \sim 1,$$

where  $\widehat{\Phi}_\infty(z)$ ,  $\widehat{\Phi}_0(z)$ , and  $\widehat{\Phi}_1(z)$  are holomorphic and invertible in neighbourhoods of the respective points. Also,

$$(3-68) \quad \widehat{\Phi}_\infty(\infty) = I,$$

$$(3-69) \quad \widehat{\Phi}_0(0) = Y^{-1}(1) \begin{pmatrix} (-1)^{-n} & 0 \\ 0 & (-1)^{k+N+n} \end{pmatrix},$$

$$(3-70) \quad \widehat{\Phi}_1(1) = Y^{-1}(1)Y(0)e^{\frac{x}{2}\sigma_3}.$$

Comparing (3-65)–(3-67) with (3-14)–(3-16) we see that, in our case, the formal monodromy exponents  $\theta_\infty$ ,  $\theta_0$ , and  $\theta_1$  are

$$(3-71) \quad \theta_\infty = -2k, \quad \theta_0 = -2n + 3k + N, \quad \theta_1 = -N - k.$$

Note that, simultaneously, these equations determine the diagonal part of the sum of the matrices  $A_0$  and  $A_1$  and also their spectrums. Indeed, from (3-65) we have that

$$(3-72) \quad \text{diag}(A_0 + A_1) = k\sigma_3,$$

and from (3-62)–(3-63) or (3-66)–(3-67) we have that

$$(3-73) \quad \text{Spect } A_0 = \left\{ \pm \frac{\theta_0}{2} \right\} \equiv \left\{ \pm \left( -n + \frac{3k+N}{2} \right) \right\},$$

and

$$(3-74) \quad \text{Spect } A_1 = \left\{ \pm \frac{\theta_1}{2} \right\} \equiv \left\{ \pm \left( \frac{N+k}{2} \right) \right\}.$$

The last three relations mean that, with  $k, n, N$  fixed, we can parameterise  $A_0$  and  $A_1$  by just *three* parameters. We denote them  $w, y, u$  and, following [27], the matrices  $A_0$  and  $A_1$  can be parameterised in the form (3-4) and (3-5). Using the general identity<sup>1</sup>

$$(3-75) \quad A_0 + A_1 = k\sigma_3 + xB_0,$$

we obtain the expression for  $B_0$  in (3-6).

As shown in [27], the compatibility condition of (3-2)–(3-3) implies that the parameters  $w, y, u$  become functions of  $x$  and they satisfy the  $3 \times 3$  system of first-order ordinary differential equations (3-8)–(3-10) with parameters

$$(3-76) \quad \alpha = \frac{1}{2} \left( \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right)^2 = \frac{1}{2} (N - n + k)^2,$$

$$(3-77) \quad \beta = -\frac{1}{2} \left( \frac{\theta_0 - \theta_1 - \theta_\infty}{2} \right)^2 = \frac{1}{2} (N - n + 3k)^2,$$

$$(3-78) \quad \gamma = 1 - \theta_0 - \theta_1 = 1 + 2n - 2k.$$

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<sup>1</sup> To derive this identity one substitutes the expansion

$$\Phi(z) \equiv \widehat{\Phi}_\infty(z) e^{\frac{x}{2}\sigma_3} z^{k\sigma_3} = \left( I + \frac{\phi_1}{z} + \dots \right) e^{\frac{x}{2}\sigma_3} z^{k\sigma_3}$$

into the Lax pair system (3-2)–(3-3). This leads to the following formulae for the matrix coefficients  $A_{0,1}$  and  $B_0$  in terms of the same matrix coefficient  $\phi_1$ :

$$A_0 + A_1 = k\sigma_3 + \frac{x}{2}[\phi_1, \sigma_3] \quad \text{and} \quad B_0 = \frac{1}{2}[\phi_1, \sigma_3].$$

Identity (3-75) follows.

Let us now obtain the formula for the Hankel determinant  $H_n[w_0]$  in terms of the functions  $y(x)$  and  $w(x)$ . We use the relation (3-19) that in our case takes the form

$$(3-79) \quad \begin{aligned} (\phi_1)_{11} &\equiv -\mathcal{H} \\ &= \frac{1}{x} \left( w - \frac{1}{y}(w-n) \right) \left( w - 2n + 3k + N - y(w + N + k - n) \right) + w + \frac{N + k - 2n}{2}. \end{aligned}$$

We have the following lemma.

**Lemma 1.** *The following relation between  $H_n[w_0]$  and  $\phi_1$  holds:*

$$(3-80) \quad \frac{d}{dx} \log H_n[w_0] = -\phi_{1,11} + \frac{N + k}{2}.$$

Lemma 1 is proven in Appendix A.

Combining (3-79) and Lemma 1, we arrive at the following expression for  $\frac{d}{dx} \log H_n[w_0]$  in terms of the Painlevé V function  $y(x)$ :

$$(3-81) \quad \frac{d}{dx} \log H_n[w_0] = \mathcal{H} + \frac{N + k}{2},$$

observing that  $w$  contained in  $\mathcal{H}$  can be expressed as a function of  $y$  and its first derivative using (3-9). In what follows we will give some details of the asymptotic expansion of  $\frac{d}{dx} \log H_n[w_0]$  by introducing the sigma function (3-20),

$$(3-82) \quad \sigma_n(x) := x \frac{d}{dx} \log H_n[w_0] - nx - n(N + k - n).$$

The  $\sigma_n$  function satisfies the equation (3-21) with parameters (3-71), namely

$$(3-83) \quad \begin{aligned} \left( x \frac{d^2 \sigma_n}{dx^2} \right)^2 &= \left( \sigma_n - x \frac{d\sigma_n}{dx} + 2 \left( \frac{d\sigma_n}{dx} \right)^2 + (4n - 4k - 2N) \frac{d\sigma_n}{dx} \right)^2 \\ &\quad - 4 \frac{d\sigma_n}{dx} \left( n - k - N + \frac{d\sigma_n}{dx} \right) \left( 2n - 3k - N + \frac{d\sigma_n}{dx} \right) \left( n + \frac{d\sigma_n}{dx} \right), \end{aligned}$$

and for the particular case  $n = k$  one obtains the equation in (1-8).

Recalling formula (3-26) for the Laguerre determinant we are studying, we have now arrived at the representation of the determinant in terms of a special solution of the fifth Painlevé equation (3-83). It remains to determine the  $\mathcal{O}(x)$  term in the asymptotic expansion of  $\sigma_k(x)$  as  $x \rightarrow 0$ .

**Lemma 2.** *The solution of the  $\sigma$ -Painlevé V equation (1-8) appearing in (1-12) has the asymptotic expansion*

$$(3-84) \quad \sigma_k(x) = -Nk + \frac{N}{2}x + \sum_{j=1}^k \alpha_{2j} x^{2j} + \mathcal{O}(x^{2k+1}) \quad \text{as } x \rightarrow 0,$$

where the coefficients  $\alpha_{2j}$ ,  $j = 1, \dots, k$ , are uniquely determined recursively from the equation (1-8).

Note that the odd-power coefficients  $\alpha_{2j+1}$  will generically be non-zero for  $j \geq k$ .

*Proof.* The function  $\sigma_k$  satisfies the equation (1-8), namely

$$(3-85) \quad \begin{aligned} \left( x \frac{d^2 \sigma_k}{dx^2} \right)^2 &= \left( \sigma_k - x \frac{d\sigma_k}{dx} + 2 \left( \frac{d\sigma_k}{dx} \right)^2 - 2N \frac{d\sigma_k}{dx} \right)^2 \\ &\quad - 4 \frac{d\sigma_k}{dx} \left( -N + \frac{d\sigma_k}{dx} \right) \left( -k - N + \frac{d\sigma_k}{dx} \right) \left( k + \frac{d\sigma_k}{dx} \right), \end{aligned}$$

with the initial data

$$(3-86) \quad \sigma_k(0) = -Nk, \quad \sigma'_k(0) = \frac{N}{2}$$

(we recall prime denotes the derivative with respect to  $x$ ). It is important to notice that equation (3-85) is degenerate at  $x = 0$  and the Cauchy-Kovalevskaya theorem is not applicable to the initial value problem (3-85)–(3-86). Moreover, at  $x = 0$  the term with  $\sigma''_k(0)$  vanishes and we get a relation between  $\sigma'_k(0)$  and  $\sigma_k(0)$ . Substituting  $\sigma_k(0) = -Nk$  into equation (3-85) and assuming  $k \neq 0$ , we obtain that  $\sigma'_k(0)$ , hence the second initial condition in (3-86) is automatically satisfied for any solution  $\sigma_k(x)$  of equation (3-85) with  $\sigma_k(0) = -Nk$ ,  $k \neq 0$ . To determine the higher-order terms in the asymptotic expansion of the function  $\sigma_k(x)$  as  $x \rightarrow 0$ , we introduce the function  $\tilde{\sigma}(x)$  such that

$$(3-87) \quad \sigma_k(x) = -Nk + \frac{Nx}{2} + \tilde{\sigma}(x).$$

If we substitute the above expression in (1-8), then we obtain the equation

$$(3-88) \quad \left(x \frac{d^2 \tilde{\sigma}}{dx^2}\right)^2 = -4x \left(\frac{d\tilde{\sigma}}{dx}\right)^3 + (4k^2 + x^2 + 4\tilde{\sigma}) \left(\frac{d\tilde{\sigma}}{dx}\right)^2 + x(N^2 + 2Nk - 2\tilde{\sigma}) \frac{d\tilde{\sigma}}{dx} + (\tilde{\sigma} - N(N + 2k))\tilde{\sigma}$$

with the initial data

$$(3-89) \quad \tilde{\sigma}(0) = 0, \quad \tilde{\sigma}'(0) = 0.$$

Equation (3-88) is degenerate at  $x = 0$  and the Cauchy-Kovalevskaya theorem is not applicable here. In fact, the initial value problem (3-88)–(3-89) has a trivial solution,  $\tilde{\sigma} = 0$ , and a nontrivial solution. We are looking for a nontrivial solution to equation (3-88) as a power series,

$$(3-90) \quad \tilde{\sigma}(x) = \sum_{j=2}^{\infty} \alpha_j x^j,$$

and we find recursively that

$$(3-91) \quad \alpha_2(16k^2\alpha_2 + N^2 + 2Nk - 4\alpha_2) = 0, \quad \alpha_2 \neq 0 \implies \alpha_2 = -\frac{N(N + 2k)}{4(4k^2 - 1)},$$

$$(3-92) \quad \frac{4\alpha_3 N(N + 2k)(k^2 - 1)}{4k^2 - 1} = 0, \quad k \neq 1 \implies \alpha_3 = 0,$$

$$(3-93) \quad \alpha_4 = \frac{N(N + 2k)(2N + 2k - 1)(2N + 2k + 1)}{16(4k^2 - 1)^2(4k^2 - 9)},$$

$$(3-94) \quad \frac{4\alpha_5 N(N + 2k)(k^2 - 4)}{4k^2 - 1} = 0, \quad k \neq 2 \implies \alpha_5 = 0,$$

$$(3-95) \quad \alpha_6 = -\frac{N(N + 2k)(2N + 2k - 1)(2N + 2k + 1)(6N^2 + 12Nk + 4k^2 - 1)}{32(4k^2 - 1)^3(4k^2 - 9)(4k^2 - 25)},$$

$$(3-96) \quad \frac{4\alpha_7 N(N + 2k)(k^2 - 9)}{4k^2 - 1} = 0, \quad k \neq 3 \implies \alpha_7 = 0,$$

and so on. An expression for  $\alpha_8$  is too long to be presented here. Observe that the odd coefficients  $\alpha_{2j+1}$  vanish as long as  $k > j$ . Indeed, we have the equation

$$(3-97) \quad \frac{4\alpha_{2j+1} N(N + 2k)(k^2 - j^2)}{4k^2 - 1} = 0,$$

hence

$$(3-98) \quad \alpha_{2j+1} = 0, \quad j = 1, 2, \dots, k-1,$$

which is equivalent to (4-10). For  $j = k$ , equation (3-97) does not determine the coefficient  $\alpha_{2j+1} = \alpha_{2k+1}$ . This implies that the initial value problem (3-88)–(3-89) has a one-parameter family of solutions, corresponding to different values of the coefficient  $\alpha_{2k+1}$ .  $\square$

#### 4. CALCULATING THE MOMENTS

The goal of this section is to calculate the quantity  $F_N(h, k)$  defined in (1-7) for  $k$  and  $h$  non-negative integers. As explained in the introduction,  $F_N(h, k)$  is related to the generalised Laguerre polynomials by

$$(4-1) \quad F_N(h, k) = \lim_{\epsilon \rightarrow 0} (-1)^{\frac{k(k-1)}{2}} 2^{-2h} \int_{-\infty}^{\infty} K_{2h}(\epsilon, y) e^{-N|y|} \det \left[ L_{N+k-1-(i+j)}^{(2k-1)}(-2|y|) \right]_{i,j=0,\dots,k-1} dy,$$

where  $K_{2h}(\epsilon, y)$  is defined in (1-6). It is convenient to rewrite the latter formula as

$$(4-2) \quad F_N(h, k) = (-1)^{\frac{k(k-1)}{2}} \lim_{\epsilon \rightarrow 0} 2^{-2h} \int_0^{\infty} K_{2h}(\epsilon, \frac{x}{2}) f_k(x) dx,$$

where

$$(4-3) \quad f_k(x) := (-1)^{\frac{k(k-1)}{2}} e^{-\frac{N}{2}x} \det \left[ L_{N+k-1-(i+j)}^{(2k-1)}(-x) \right]_{i,j=0,\dots,k-1}.$$

**Lemma 3.** *The following identity is satisfied:*

$$(4-4) \quad F_N(h, k) = (-1)^h f_k^{(2h)}(0), \quad h \in \mathbb{N}_0,$$

where  $f_k(x)$  is given in (4-3).

*Proof.* By (4-2),

$$(4-5) \quad \begin{aligned} F_N(h, k) &= \lim_{\epsilon \rightarrow 0} 2^{-2h} \frac{\partial^{2h}}{\partial \epsilon^{2h}} \int_0^{\infty} \frac{\epsilon f_k(x) dx}{\pi(\epsilon^2 + (\frac{x}{2})^2)} \\ &= 2 \lim_{\epsilon \rightarrow 0} \frac{\partial^{2h}}{\partial \epsilon^{2h}} \int_0^{\infty} \frac{f_k(\epsilon y) dy}{\pi(1 + y^2)} \\ &= 2 \lim_{\epsilon \rightarrow 0} \int_0^{\infty} \frac{y^{2h} f_k^{(2h)}(\epsilon y) dy}{\pi(1 + y^2)}. \end{aligned}$$

We have that

$$(4-6) \quad y^{2h} = (y^2 + 1) \left( y^{2h-2} - y^{2h-4} + \dots + (-1)^{h+1} \right) + (-1)^h,$$

hence

$$(4-7) \quad \int_0^{\infty} \frac{y^{2h} f_k^{(2h)}(\epsilon y) dy}{1 + y^2} = \int_0^{\infty} \left( y^{2h-2} - y^{2h-4} + \dots + (-1)^{h+1} + \frac{(-1)^h}{y^2 + 1} \right) f_k^{(2h)}(\epsilon y) dy.$$

Integrating by parts  $2j$  times, we obtain that

$$(4-8) \quad \begin{aligned} \int_0^{\infty} y^{2j} f_k^{(2h)}(\epsilon y) dy &= \epsilon^{-2j} (2j)! \int_0^{\infty} f_k^{(2h-2j)}(\epsilon y) dy \\ &= -\epsilon^{-2j-1} (2j)! f_k^{(2h-2j-1)}(0), \quad 0 \leq j \leq h-1, \end{aligned}$$

hence

$$(4-9) \quad \lim_{\epsilon \rightarrow 0} \int_0^{\infty} \frac{y^{2h} f_k^{(2h)}(\epsilon y) dy}{1 + y^2} = (-1)^h \frac{\pi}{2} f_k^{(2h)}(0) + \lim_{\epsilon \rightarrow 0} \sum_{j=0}^{h-1} (-1)^{h-j} \epsilon^{-2j-1} (2j)! f_k^{(2h-2j-1)}(0).$$



Since  $F_N(h, k)$  is finite, all terms in the latter sum vanish, so that

$$(4-10) \quad f_k^{(2h-2j-1)}(0) = 0, \quad 0 \leq j \leq h-1.$$

Thus,

$$(4-11) \quad \lim_{\epsilon \rightarrow 0} \int_0^\infty \frac{y^{2h} f_k^{(2h)}(\epsilon y) dy}{1+y^2} = (-1)^h \frac{\pi}{2} f_k^{(2h)}(0).$$

Therefore we get the statement of the lemma.  $\square$

**4.1. Evaluation of  $F_N(0, k)$ .** From (4-11) with  $h = 0$  we have that

$$(4-12) \quad \begin{aligned} F_N(0, k) &= (-1)^{\frac{k(k-1)}{2}} \det \left[ L_{N+k-1-(i+j)}^{(2k-1)}(0) \right]_{i,j=0,\dots,k-1} \\ &= (-1)^{\frac{k(k-1)}{2}} \det \left[ \frac{(N+3k-2-i-j)!}{(2k-1)!(N+k-1-i-j)!} \right]_{i,j=0,\dots,k-1} \\ &= \frac{(-1)^{\frac{k(k-1)}{2}}}{[(2k-1)!]^k} \det \left[ \frac{(N+3k-2-i-j)!}{(N+k-1-i-j)!} \right]_{i,j=0,\dots,k-1}. \end{aligned}$$

In particular,

$$(4-13) \quad F_N(0, 1) = N+1,$$

$$(4-14) \quad \begin{aligned} F_N(0, 2) &= -\frac{1}{36} \det \begin{bmatrix} (N+2)(N+3)(N+4) & (N+1)(N+2)(N+3) \\ (N+1)(N+2)(N+3) & N(N+1)(N+2) \end{bmatrix} \\ &= -\frac{1}{36} (N+1)(N+2)^2(N+3) \det \begin{bmatrix} N+4 & N+1 \\ N+3 & N \end{bmatrix} \\ &= \frac{1}{12} (N+1)(N+2)^2(N+3), \end{aligned}$$

$$(4-15) \quad \begin{aligned} F_N(0, 3) &= -\frac{1}{(5!)^3} (N+1)(N+2)^2(N+3)^3(N+4)^2(N+5) \\ &\quad \times \det \begin{bmatrix} (N+6)(N+7) & (N+2)(N+6) & (N+1)(N+2) \\ (N+5)(N+6) & (N+1)(N+5) & N(N+1) \\ (N+4)(N+5) & N(N+4) & (N-1)N \end{bmatrix} \\ &= \frac{1}{3 \cdot 4! \cdot 5!} (N+1)(N+2)^2(N+3)^3(N+4)^2(N+5), \end{aligned}$$

and so on. In general,

$$(4-16) \quad \begin{aligned} F_N(0, k) &= C_k (N+1)(N+2)^2 \cdots (N+k-1)^{k-1} (N+k)^k \\ &\quad \times (N+k+1)^{k-1} \cdots (N+2k-2)^2 (N+2k-1). \end{aligned}$$

To find the constant  $C_k$ , consider  $N = 0$ :

$$(4-17) \quad F_0(0, k) = C_k \cdot 1 \cdot 2^2 \cdots (k-1)^{k-1} k^k (k+1)^{k-1} \cdots (2k-2)^2 (2k-1).$$

On the other hand, by (4-12),

$$(4-18) \quad F_0(0, k) = \frac{(-1)^{\frac{k(k-1)}{2}}}{[(2k-1)!]^k} \det \left[ \frac{(3k-2-i-j)!}{(k-1-i-j)!} \right]_{i,j=0,\dots,k-1} = 1,$$

hence

$$(4-19) \quad C_k = \frac{1}{1 \cdot 2^2 \cdots (k-1)^{k-1} k^k (k+1)^{k-1} \cdots (2k-2)^2 (2k-1)}.$$

Thus, we have the formula in [32],

$$\begin{aligned}
 F_N(0, k) &= \frac{G(N+2k+1)G(N+1)G(k+1)^2}{G(N+k+1)^2G(2k+1)} \\
 (4-20) \quad &= \frac{1}{1 \cdot 2^2 \cdots (k-1)^{k-1} k^k (k+1)^{k-1} \cdots (2k-2)^2 (2k-1)} \\
 &\quad \times (N+1)(N+2)^2 \cdots (N+k-1)^{k-1} (N+k)^k \\
 &\quad \times (N+k+1)^{k-1} \cdots (N+2k-2)^2 (N+2k-1),
 \end{aligned}$$

where  $G(z)$  is the Barnes  $G$ -function (see Appendix C). In particular, this implies that

$$(4-21) \quad F(0, k) = \lim_{N \rightarrow \infty} \frac{F_N(0, k)}{N^{k^2}} = \frac{G(k+1)^2}{G(2k+1)},$$

as shown in [32].

**4.2. Evaluation of  $F_N(h, k)$ .** In order to evaluate the function  $F_N(h, k)$  we use the following identities between the Hankel determinant  $H_k$  in (1-10) and the function  $f_k(x)$  defined in (4-3),

$$(4-22) \quad H_k(x) = (2\pi i)^k e^{kx} \det \left[ L_{N+k-1-(i+j)}^{(2k-1)}(-x) \right]_{i,j=0,\dots,k-1} = (2\pi i)^k e^{(k+\frac{N}{2})x} (-1)^{\frac{k(k-1)}{2}+h} f_k(x).$$

Furthermore, the function  $H_k(x)$  is related to a solution of the  $\sigma$ -Painlevé V equation (1-8)

$$(4-23) \quad \frac{d}{dx} \log H_k = \frac{\sigma_k(x) + kx + Nk}{x},$$

so that combining the above two relations we obtain

$$(4-24) \quad x \frac{d}{dx} \log f_k(x) = \sigma_k(x) - \frac{N}{2}x + Nk = \tilde{\sigma}(x) = \sum_{j=2}^{\infty} \alpha_j x^j,$$

where  $\tilde{\sigma}(x)$  has been defined in (3-87) and the first few coefficient  $\alpha_j$  have been evaluated in (3-91)–(3-96). Integrating the above equation, we obtain that

$$\begin{aligned}
 (4-25) \quad f_k(x) &= f_k(0) \exp \left( \sum_{j=2}^{\infty} \frac{\alpha_j x^j}{j} \right) = f_k(0) \left[ \sum_{j=0}^k \beta_{2j} x^{2j} + \sum_{j=2k+1}^{\infty} \beta_j x^j \right] \\
 &= \left[ 1 + \frac{1}{2} \alpha_2 x^2 + \left( \frac{\alpha_4}{4} + \frac{\alpha_2^2}{8} \right) x^4 + \left( \frac{\alpha_6}{6} + \frac{\alpha_2 \alpha_4}{8} + \frac{\alpha_2^3}{48} \right) x^6 \right. \\
 &\quad \left. + \left( \alpha_8 + \alpha_2 \alpha_6 + \frac{1}{2} \alpha_4 \alpha_4^2 + \frac{1}{2} \alpha_4^2 + \frac{1}{24} \alpha_2^4 \right) x^8 + \cdots \right]
 \end{aligned}$$

where the coefficients  $\beta_j$  are obtained from the above expansion and using the explicit expression of  $\alpha_j$  in (3-91)–(3-96) as

$$(4-26) \quad \beta_2 = \frac{1}{2} \alpha_2 = \frac{N(N+2k)}{4(4k^2-1)}, \quad \beta_4 = \frac{\alpha_4}{4} + \frac{\alpha_2^2}{8} = \frac{N(N+2k)(N^2+2kN+2)}{128(4k^2-1)(4k^2-9)}, \quad \dots$$

and so on. Observe that by (3-96), all odd powers  $x^{2j+1}$  will be missing on the right-hand side as long as  $j < k$ . In other words,

$$(4-27) \quad f_k^{2j+1}(0) = 0, \quad j = 0, 1, \dots, k-1,$$

which is equivalent to equation (4-10). From Lemma 3 we have the relation

$$(4-28) \quad F_N(h, k) = (-1)^h f_k^{(2h)}(0)$$

which implies, using the explicit expressions of  $\alpha_j$  in (3-91)–(3-96),

$$\begin{aligned}
 F_N(1, k) &= \frac{N(N+2k)}{4(4k^2-1)} F_N(0, k), \\
 (4-29) \quad F_N(2, k) &= \frac{3N(N+2k)(N^2+2kN+2)}{16(4k^2-1)(4k^2-9)} F_N(0, k), \\
 F_N(3, k) &= -\frac{15}{64} \frac{N(2k+N)}{(4k^2-25)(k^2-9)(4k^2-1)^2} (-16+64k^2+20kN+48k^3N+10N^2 \\
 &\quad -12k^2N^2+16k^4N^2-36kN^3+16k^3N^3-9N^4+4k^2N^4) F_N(0, k),
 \end{aligned}$$

and so on. A formula for  $F_N(4, k)$  is too long to be presented here.

**4.3. Scaling limit of Painlevé V as  $N \rightarrow \infty$  and the second main result.** From equations (3-91), (3-93), and (3-95) we find

$$\begin{aligned}
 (4-30) \quad \xi_2 &:= \frac{\alpha_2}{N^2} = -\frac{N(N+2k)}{4(4k^2-1)N^2} \xrightarrow{N \rightarrow \infty} -\frac{1}{4(4k^2-1)}, \\
 \xi_4 &:= \frac{\alpha_4}{N^4} \xrightarrow{N \rightarrow \infty} \frac{1}{4(4k^2-1)^2(4k^2-9)}, \\
 \xi_6 &:= \frac{\alpha_6}{N^6} \xrightarrow{N \rightarrow \infty} -\frac{3}{4(4k^2-1)^3(4k^2-9)(4k^2-25)},
 \end{aligned}$$

and so forth. Define, therefore, the function

$$(4-31) \quad \xi(t) = \sum_{j=2}^{\infty} \xi_j t^j := \sum_{j=0}^{\infty} \frac{\alpha_j t^j}{N^j} = \tilde{\sigma} \left( \frac{t}{N} \right).$$

From equation (3-88) we obtain that  $\xi(t)$  satisfies the equation

$$\begin{aligned}
 (4-32) \quad \left( t \frac{d^2 \xi}{dt^2} \right)^2 &= -4t \left( \frac{d\xi}{dt} \right)^3 + (4k^2 + 4\xi + N^{-2}t^2) \left( \frac{d\xi}{dt} \right)^2 + t(1 + 2N^{-1}k - 2N^{-2}\xi) \frac{d\xi}{dt} \\
 &\quad - (1 + 2N^{-1}k - N^{-2}\xi)\xi.
 \end{aligned}$$

In the limit  $N \rightarrow \infty$  it reduces to the equation

$$(4-33) \quad \left( t \frac{d^2 \xi}{dt^2} \right)^2 = -4t \left( \frac{d\xi}{dt} \right)^3 + (4k^2 + 4\xi) \left( \frac{d\xi}{dt} \right)^2 + t \frac{d\xi}{dt} - \xi.$$

This equation can be identified as a special case of the  $\sigma$ -Painlevé III' equation

$$(4-34) \quad \left( t \frac{d^2 v}{dt^2} \right)^2 + \theta_0 \theta_{\infty} \frac{dv}{dt} - \left( 4 \left( \frac{dv}{dt} \right)^2 - 1 \right) \left( v - t \frac{dv}{dt} \right) - \frac{1}{4}(\theta_0^2 + \theta_{\infty}^2) = 0$$

considered by Okamoto [43, Proposition 1.7] by choosing  $\xi = v - k^2$ ,  $\theta_0 = 0$ , and  $\theta_{\infty} = -2k$ . The initial data for (4-33) are

$$(4-35) \quad \xi(0) = 0, \quad \xi'(0) = 0.$$

As before, the condition on  $\xi'(0)$  follows from  $\xi(0) = 0$  and equation (4-32). We are looking for a solution

$$(4-36) \quad \xi(t) = \sum_{j=2}^{\infty} \xi_j t^j$$

to equation (4-33) such that

$$(4-37) \quad \xi_2 \neq 0.$$

We have that

$$(4-38) \quad \xi_{2j+1} = 0, \quad j = 1, 2, \dots, k-1.$$

The even coefficients,  $\xi_{2j}$ ,  $j = 1, 2, \dots, k$ , can be found recursively from equation (4-33). From equations (4-21), (4-25), and (4-28) we can consider the rescaled quantity

$$(4-39) \quad \frac{F_N(h, k)}{N^{k^2+2h}} = (-1)^h \frac{d^{2h}}{dt^{2h}} f_k \left( \frac{t}{N} \right) \Big|_{t=0}.$$

We arrive at our second main result.

**Theorem 2.** *The limit*

$$(4-40) \quad F(h, k) = \lim_{N \rightarrow \infty} \frac{F_N(h, k)}{N^{k^2+2h}}$$

*exists, and it is given by the formula*

$$(4-41) \quad F(h, k) = (-1)^h F(0, k) \frac{d^{2h}}{dt^{2h}} \left[ \exp \left( \sum_{j=2}^{\infty} \frac{\xi_j t^j}{j} \right) \right] \Big|_{t=0},$$

where  $F(0, k)$  is given in (4-21) and the power series (4-36) solves equation (4-33).

It is noteworthy that equation (4-33) is closely related to the  $\sigma$ -Painlevé III (as opposed to III') equation. Namely, let

$$(4-42) \quad \sigma_{III}(s) := 2\xi(s^2) + k^2.$$

Then  $\sigma_{III}(s)$  solves the  $\sigma$ -Painlevé III equation (cf. [27, (C.29)]),

$$(4-43) \quad \left( s \frac{d^2 \sigma_{III}}{ds^2} - \frac{d\sigma_{III}}{ds} \right)^2 = 4 \left( 2\sigma_{III} - s \frac{d\sigma_{III}}{ds} \right) \left( \left( \frac{d\sigma_{III}}{ds} \right)^2 - 4s^2 \right) + 2(\theta_0^2 + \theta_\infty^2) \left( \left( \frac{d\sigma_{III}}{ds} \right)^2 + 4s^2 \right) - 16\theta_0\theta_\infty s \frac{d\sigma_{III}}{ds},$$

for  $\theta_0 = 0$  and  $\theta_\infty = -2k$  with the initial conditions

$$(4-44) \quad \sigma_{III}(0) = k^2, \quad \sigma'_{III}(0) = 0.$$

We write the first few values of  $F(h, k)$ :

$$(4-45) \quad \begin{aligned} F(1, k) &= \frac{F(0, k)}{4(4k^2 - 1)}, \\ F(2, k) &= \frac{3F(0, k)}{16(4k^2 - 1)(4k^2 - 9)}, \\ F(3, k) &= \frac{15F(0, k)}{64(4k^2 - 1)^2(4k^2 - 25)}, \\ F(4, k) &= \frac{105(4k^2 - 33)F(0, k)}{256(4k^2 - 1)^2(4k^2 - 9)(4k^2 - 25)(4k^2 - 49)}, \\ F(5, k) &= \frac{925(16k^4 - 360k^2 + 1497)F(0, k)}{1024(4k^2 - 1)^2(4k^2 - 9)^2(4k^2 - 25)(4k^2 - 49)(4k^2 - 81)}, \end{aligned}$$

and so on. In [12] an explicit formula for the above expansions has been derived and it takes the form

$$(4-46) \quad F(h, k) = \frac{(2h)!}{h!2^{3h}} F(0, k) \frac{\tilde{X}_{2h}(2k)}{Y_{2h}(2k)},$$

where the polynomials  $\tilde{X}_{2h}(s)$  are obtained in a combinatorial way and we report the first few (see Table 4 in [12]):

$$(4-47) \quad \tilde{X}_2(s) = 1, \quad \tilde{X}_4(s) = 1, \quad \tilde{X}_6(s) = s^2 - 9, \quad \tilde{X}_8(s) = s^2 - 33, \quad \tilde{X}_{10}(s) = s^4 - 90s^2 + 1497,$$

and the polynomial  $Y_{2h}(s)$  is given by the expression

$$(4-48) \quad Y_r(s) = \prod_{\substack{1 \leq a \leq r-1 \\ a \text{ odd}}} (s^2 - a^2)^{\eta_a(r)}, \quad \eta_a(r) = \left\lfloor \frac{-a + \sqrt{a^2 + 4r}}{2} \right\rfloor,$$

where the symbol  $\lfloor z \rfloor$  denotes the integer part of  $z$ . We have

$$(4-49) \quad \begin{aligned} Y_2(s) &= s^2 - 1, \quad Y_4(s) = (s^2 - 9)(s^2 - 1), \quad Y_6(s) = (s^2 - 1)^2(s^2 - 9)(s^2 - 25), \\ Y_8(s) &= (s^2 - 1)^2(s^2 - 9)(s^2 - 25)(s^2 - 49), \\ Y_{10}(s) &= (s^2 - 1)^2(s^2 - 9)^2(s^2 - 25)(s^2 - 49)(s^2 - 81). \end{aligned}$$

Combining the above expressions we can verify that the formula (4-46) reproduces the terms obtained in (4-45).

## 5. SCALING LIMIT OF THE RIEMANN-HILBERT PROBLEM AS $N \rightarrow \infty$

In this section we will supplement the result of the previous section by performing the large- $N$  scaling limit directly in the  $X$ -Riemann-Hilbert problem (3-38)–(3-40). We start by changing the jump contour of the problem.

Let  $\Gamma_N$  be the positively oriented circle of radius  $N$  centered at  $z = 0$  and let us pass from the original function  $X \equiv X_N(z, x; k, n)$  to the new matrix valued function  $\tilde{X} \equiv \tilde{X}_N(z, x; k, n)$  according to the following rule.

- For all  $z$  inside the small circle  $\Gamma$  and outside the big circle  $\Gamma_N$ , we put

$$(5-1) \quad \tilde{X}_N(z) := X_N(z).$$

- For all  $z$  between the circles we define

$$(5-2) \quad \tilde{X}_N(z) := X_N(z) \begin{pmatrix} 1 & e^{xz} \frac{z^{3k+N}}{(z-1)^{N+k}} \\ 0 & 1 \end{pmatrix}.$$

The new Riemann-Hilbert problem reads

$$(5-3) \quad \tilde{X}_N(z) \in \mathcal{H}(\mathbb{CP}^1 \setminus (\Gamma_N \cup \{0\})),$$

$$(5-4) \quad \tilde{X}_{N+}(z) = \tilde{X}_{N-}(z) \begin{pmatrix} 1 & e^{xz} \frac{z^{3k+N}}{(z-1)^{N+k}} \\ 0 & 1 \end{pmatrix}, \quad z \in \Gamma_N,$$

$$(5-5) \quad \tilde{X}_N(z) \sim (I + \mathcal{O}(z) + \dots) (-z)^{-n\sigma_3}, \quad z \rightarrow 0.$$

The  $\tilde{X}$ -Riemann-Hilbert problem is ready for the large- $N$  scaling limit. Put

$$(5-6) \quad Z_N(z, t; k, n) := \tilde{X}_N\left(zN, \frac{t}{N}; k, n\right) N^{n\sigma_3}.$$

Then, the Riemann-Hilbert problem (5-3)–(5-5) transforms to the following Riemann-Hilbert problem for the function  $Z_N$  posed on the unit circle  $\Gamma_1 := \{z : |z| = 1\}$ :

$$(5-7) \quad Z_N(z) \in \mathcal{H}(\mathbb{CP}^1 \setminus (\Gamma_1 \cup \{0\})),$$

$$(5-8) \quad Z_{N+}(z) = Z_{N-}(z) \begin{pmatrix} 1 & e^{tz} \frac{z^{2k} N^{2(k-n)}}{(1 - \frac{1}{zN})^{N+k}} \\ 0 & 1 \end{pmatrix}, \quad z \in \Gamma_1,$$

$$(5-9) \quad Z_N(z) \sim (I + \mathcal{O}(z) + \dots) (-z)^{-n\sigma_3}, \quad z \rightarrow 0.$$

Assume now, and this is the case of our main concern, that

$$n = k.$$

Then, as  $N \rightarrow \infty$  the new jump matrix converges to the matrix

$$\begin{pmatrix} 1 & z^{2k} e^{tz + \frac{1}{z}} \\ 0 & 1 \end{pmatrix}$$

uniformly for  $z \in \Gamma_1$  and  $t \in K$ , where  $K$  is a compact set in  $\mathbb{C}$ . By standard Riemann-Hilbert arguments, this implies the convergence of the function  $Z_N(z, t)$  to the function  $Z(z, t)$  satisfying Riemann-Hilbert problem

$$(5-10) \quad Z(z) \in \mathcal{H}(\mathbb{CP}^1 \setminus (\Gamma_1 \cup \{0\})),$$

$$(5-11) \quad Z_+(z) = Z_-(z) \begin{pmatrix} 1 & z^{2k} e^{tz + \frac{1}{z}} \\ 0 & 1 \end{pmatrix}, \quad z \in \Gamma_1,$$

$$(5-12) \quad Z(z) = (I + \mathcal{O}(z)) (-z)^{-k\sigma_3}, \quad z \rightarrow 0.$$

In fact, the estimate

$$(5-13) \quad Z_N(z, t) = \left( I + \mathcal{O} \left( \frac{1}{N(1 + |z|)} \right) \right) Z(z, t), \quad N \rightarrow \infty,$$

holds uniformly for all  $z \in \mathbb{CP}^1$  and  $t \in K$ , where  $K$  is a compact set in  $\mathbb{C}$ .

Denote by  $\phi_1^{Z_N} \equiv \phi_1^{Z_N}(t)$  and  $\phi_1^Z \equiv \phi_1^Z(t)$  the first matrix coefficients in the expansions near  $z = \infty$  of the functions  $Z_N(z)$  and  $Z(z)$ , respectively:

$$(5-14) \quad Z_N(z) = Z_N(\infty) \left( I + \frac{\phi_1^{Z_N}}{z} + \dots \right), \quad z \rightarrow \infty$$

and

$$(5-15) \quad Z(z) = Z(\infty) \left( I + \frac{\phi_1^Z}{z} + \dots \right), \quad z \rightarrow \infty.$$

Estimate (5-13) implies that

$$(5-16) \quad \phi_1^{Z_N}(t) \xrightarrow{N \rightarrow \infty} \phi_1^Z(t)$$

uniformly for all  $t \in K$ , where  $K$  is a compact set in  $\mathbb{C}$ . At the same time, recalling the connection of  $Z_N(z)$  with the function  $X(z)$  and the connection of the latter with  $\Phi(z) \equiv \Phi_N(z, t; k)$ , we arrive at the relation

$$(5-17) \quad \phi_{1,11} \left( \frac{t}{N} \right) = N \phi_{1,11}^{Z_N}(t) + \frac{N+k}{2}.$$

From this relation and (5-16) we have

$$(5-18) \quad \frac{1}{N} \phi_{1,11} \left( \frac{t}{N} \right) \xrightarrow{N \rightarrow \infty} \phi_{1,11}^Z(t) + \frac{1}{2}.$$

Together with Lemma 1, the last limit allows us to find the large- $N$  scaling limit of the Hankel determinant  $H_k[w_0] \equiv H(x)$  in terms of the solution of the  $Z$ -Riemann-Hilbert problem:

$$(5-19) \quad \frac{d}{dt} \log H_k \left( \frac{t}{N} \right) \xrightarrow{N \rightarrow \infty} -\phi_{1,11}^Z(t).$$

In the previous subsection we have already connected this limit with the special solution  $\xi(z)$  of the third Painlevé equation (4-33). Let us show how the same result can be derived from the  $Z$ -Riemann-Hilbert problem (5-10)–(5-12).

Similar to the finite- $N$  case, we introduce the function (cf. (3-43)),

$$(5-20) \quad \Phi^{(III)}(z, s) := s^{k\sigma_3} Z^{-1}(\infty) Z\left(\frac{z}{s}, s^2\right) \left(\frac{z}{s}\right)^{k\sigma_3} e^{\frac{s}{2}(z+\frac{1}{z})\sigma_3}.$$

The Riemann-Hilbert problem (5-10)–(5-12) in terms of  $\Phi^{(III)}(z, t)$  reads

$$(5-21) \quad \Phi^{(III)}(z) \in \mathcal{H}(\mathbb{C} \setminus (\Gamma_1 \cup \{0\})),$$

$$(5-22) \quad \Phi_+^{(III)}(z) = \Phi_-^{(III)}(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad z \in \Gamma_1,$$

$$(5-23) \quad \Phi^{(III)}(z) = P_0 \left( I + \mathcal{O}(z) \right) e^{\frac{s}{2z}\sigma_3}, \quad z \rightarrow 0,$$

$$(5-24) \quad \Phi^{(III)}(z) = \left( I + \frac{\phi_1^{(III)}}{z} + \dots \right) z^{k\sigma_3} e^{\frac{sz}{2}\sigma_3}, \quad z \rightarrow \infty,$$

where

$$(5-25) \quad \phi_1^{(III)}(s) = s s^{k\sigma_3} \phi_1^Z(s^2) s^{-k\sigma_3} + \frac{s}{2}\sigma_3$$

and

$$(5-26) \quad P_0 := (-s)^{k\sigma_3} Z^{-1}(\infty).$$

Repeating now the same standard argument based on Liouville's theorem as in §3 which led us to the Painlevé V Lax pair (3-53)–(3-54), we arrive at the following Lax pair for the function  $\Phi^{(III)}(z, s)$ :

$$(5-27) \quad \frac{\partial \Phi^{(III)}}{\partial z} = \left( \frac{s}{2}\sigma_3 + \frac{A_0}{z} + \frac{A_{-1}}{z^2} \right) \Phi^{(III)}(z, s),$$

$$(5-28) \quad \frac{\partial \Phi^{(III)}}{\partial s} = \left( \frac{z}{2}\sigma_3 + B_0 + \frac{B_{-1}}{z} \right) \Phi^{(III)}(z, s),$$

which is *exactly* the Jimbo-Miwa Lax pair for the Painlevé III equation (see (C.18), (C.19) of [27]). A comparison of the asymptotics (5-23) and (5-24) with the formulae (C.26) and (C.25) of [27], respectively, shows that the Jimbo-Miwa formal monodromy parameters  $\theta_\infty$  and  $\theta_0$  are

$$(5-29) \quad \theta_\infty = -2k, \quad \theta_0 = 0.$$

Following [27], we parameterise the matrix coefficients  $A_0, A_{-1}$  and  $B_0, B_{-1}$  by the functional parameters  $y, w, u, v$  according to the equations ( $z^{JM} \equiv w, t^{JM} = s$ )

$$(5-30) \quad A_0 := \begin{pmatrix} k & u \\ v & -k \end{pmatrix} = sB_0 + k\sigma_3,$$

$$(5-31) \quad A_1 = \begin{pmatrix} w - \frac{s}{2} & \frac{u}{y} \\ -\frac{yw}{u}(w - s) & -w + \frac{s}{2} \end{pmatrix} = -sB_{-1}.$$

We will also need the following formula for the matrix coefficient  $\phi_1^{(III)}$  which can be obtained by the substitution of the expansion (5-24) into the equation (5-27):

$$(5-32) \quad \phi_1^{(III)} = \begin{pmatrix} -s^{-1}uv - w + \frac{s}{2} & -s^{-1}u \\ s^{-1}v & s^{-1}uv + w - \frac{s}{2} \end{pmatrix}.$$



The compatibility condition of equations (5-27) and (5-28) yields the following deformation equations on  $y(s)$  and  $w(s)$  (cf. [27, (C.23)]):

$$(5-33) \quad s \frac{dy}{ds} = 4wy^2 - 2sy^2 + (2\theta_\infty - 1)y + 2s$$

and

$$(5-34) \quad s \frac{dw}{ds} = -4yw^2 - 2sy^2 + (4sy - 2\theta_\infty + 1)w + (\theta_0 + \theta_\infty)s.$$

We recall that in our case  $\theta_0 = 0$  and  $\theta_\infty = -2k$ . In turn, this system implies a single second-order ODE, the Painlevé III (as opposed to III') equation for the function  $y(s)$  (cf. [27, (C.23)] or [43]),

$$(5-35) \quad \frac{d^2y}{ds^2} = \frac{1}{y} \left( \frac{dy}{ds} \right)^2 - \frac{1}{s} \frac{dy}{ds} + \frac{1}{s}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y},$$

with

$$\alpha = 4\theta_0 = 0, \quad \beta = 4(1 - \theta_\infty) = 4(1 + 2k), \quad \gamma = -\delta = 4.$$

Moreover, from [27] one can extract the following addition expression for the 11-entry of the matrix coefficient  $\phi_1^{(III)}$  in the expansion (5-24) of the solution  $\Phi^{(III)}(z)$  at  $z = \infty$  (cf. (3-18)):

$$(5-36) \quad (\phi_1^{(III)})_{11} = -\frac{1}{2}\mathcal{H}_{III} + \frac{k^2}{2t},$$

where

$$(5-37) \quad \mathcal{H}_{III} \equiv \mathcal{H}_{III}(y, w; s) = \frac{1}{s} \left( 2w^2y^2 + 2w(s - sy^2 - 2ky) + 2ksy - s^2 + k^2 \right)$$

is the Hamiltonian of the dynamical system (5-33)–(5-34). Correspondingly, the tau function  $\tau_{III}(s)$  and the sigma function  $\sigma_{III}(s)$  are defined by the equations

$$(5-38) \quad \frac{d \log \tau_{III}(s)}{ds} = \mathcal{H}_{III}(y(s), w(s); s) = \frac{\sigma_{III}(s)}{s},$$

and the sigma-form of the third Painlevé equation (5-35) reads (cf. [27, (C.29)])

$$(5-39) \quad \left( s \frac{d^2\sigma_{III}}{ds^2} - \frac{d\sigma_{III}}{ds} \right)^2 = 4 \left( 2\sigma_{III} - s \frac{d\sigma_{III}}{ds} \right) \left( \left( \frac{d\sigma_{III}}{ds} \right)^2 - 4s^2 \right) + 2(\theta_0^2 + \theta_\infty^2) \left( \left( \frac{d\sigma_{III}}{ds} \right)^2 + 4s^2 \right) - 16\theta_0\theta_\infty s \frac{d\sigma_{III}}{ds}$$

with

$$(5-40) \quad \theta_0 = 0, \quad \theta_\infty = -2k.$$

Equations (5-38), (5-36), and (5-25) yield the following formula for the matrix entry  $(\phi_1^Z)_{11}$ :

$$(5-41) \quad (\phi_1^Z(t))_{11} = \frac{k^2 - \sigma_{III}(\sqrt{t})}{2t} - \frac{1}{2}.$$

Noticing that

$$\sigma_{III}(\sqrt{t}) = 2\xi(t) + k^2,$$

where  $\xi(t)$  is the solution of the  $\xi$ -form of the third Painlevé equation (4-33), we conclude that

$$(5-42) \quad (\phi_1^Z(t))_{11} = -\frac{\xi(t)}{t} - \frac{1}{2}.$$

This, together with (5-19), yields the limit formula

$$(5-43) \quad \frac{d}{dt} \log H_k \left( \frac{t}{N} \right) \xrightarrow{N \rightarrow \infty} \frac{\xi(t)}{t} + \frac{1}{2}.$$

For our principal object, the function  $f_k(x)$  (see (4-22)), we have that

$$(5-44) \quad \frac{d}{dt} \log f_k \left( \frac{t}{N} \right) \xrightarrow{N \rightarrow \infty} \frac{\xi(t)}{t}.$$

This is our second main result, i.e. Theorem 2.

It is worth noticing that the importance of the  $Z$ -Riemann-Hilbert problem lies in the fact that it can be used for the large- $t$  asymptotic analysis of the function  $\xi(t)$  which will be needed in the case of the half integer  $h$ . Indeed, in that case the expression for  $F_N(h, k)$  in terms of  $\sigma(s)$  and hence the expression for  $F(h, k)$  in terms of  $\xi(t)$  are not local – see the next section – and therefore global information about the behavior of  $\xi(t)$  is essential.

We conclude this section by making the following interesting observation concerning the  $Z$ -Riemann-Hilbert problem. Put

$$(5-45) \quad Y^{(III)}(z) := \Phi^{(III)}(z, s) e^{-\frac{s}{2}(z+\frac{1}{z})\sigma_3} = s^{k\sigma_3} Z^{-1}(\infty) Z \left( \frac{z}{s}, s^2 \right) \left( \frac{z}{s} \right)^{k\sigma_3}.$$

Then, the Riemann-Hilbert problem (5-21)–(5-24) in terms of  $Y^{(III)}(z, t)$  reads

$$(5-46) \quad Y^{(III)}(z) \in \mathcal{H}(\mathbb{C} \setminus \Gamma_1),$$

$$(5-47) \quad Y_+^{(III)}(z) = Y_-^{(III)}(z) \begin{pmatrix} 1 & e^{s(z+\frac{1}{z})} \\ 0 & 1 \end{pmatrix}, \quad z \in \Gamma_1,$$

$$(5-48) \quad Y^{(III)}(z) = \left( I + \frac{\phi_1^{YIII}}{z} + \dots \right) z^{k\sigma_3}, \quad z \rightarrow \infty,$$

where

$$(5-49) \quad \phi_1^{YIII}(s) = s s^{k\sigma_3} \phi_1^Z(s^2) s^{-k\sigma_3}.$$

This is again an example of a Riemann-Hilbert problem from the theory of Hankel determinants. The corresponding contour and weight, this time, are the unit circle  $\Gamma_1$  and the Bessel type weight

$$(5-50) \quad w_0^{(III)}(z) = e^{s(z+\frac{1}{z})},$$

respectively. In fact, the Hankel determinant associated with the problem (5-46)–(5-48) is the determinant

$$(5-51) \quad H_k^{(III)}(s) = \det \left[ \int_{\Gamma_1} z^{i+j} e^{s(z+\frac{1}{z})} dz \right]_{i,j=0,\dots,k-1} = (2\pi i)^k \det [I_{i+j+1}(2s)]_{i,j=0,\dots,k-1}.$$

The relations similar to (3-33) take the form

$$(5-52) \quad \frac{H_{k+1}^{(III)}}{H_k^{(III)}} = h_k, \quad h_k = -2\pi i (\phi_1^{YIII})_{12}, \quad \frac{1}{h_{k-1}} = -\frac{1}{2\pi i} (\phi_1^{YIII})_{21},$$

where now  $h_k := \int_{\Gamma_1} P_k^2(z) w_0^{(III)}(z) dz$  and  $P_k(z)$  are monic polynomials orthogonal on  $\Gamma_1$  with respect to the weight (5-50). These relations, as in the case of our original Laguerre-Hankel determinant  $H_n$ , can be used to prove the following analogue of Lemma 1.

**Lemma 4.** *The following relation between  $H_k^{(III)}(s)$  and  $\phi_1^{YIII}$  holds:*

$$(5-53) \quad \frac{d}{ds} \log H_k^{(III)}(s) = -2\phi_{1,11}^{YIII}(s) + \frac{k^2}{s}.$$

*Proof.* It is convenient to make yet another change-of-variable transformation of the function  $\Phi^{(III)}(z, s)$ , namely,

$$(5-54) \quad \Phi^{(III)}(z, s) \rightarrow \tilde{\Phi}^{(III)}(z, s) = s^{-k\sigma_3} \Phi^{(III)}(zs, s).$$

The motivation for this transformation is that the  $s$ -equation of the Lax pair for the function  $\tilde{\Phi}^{(III)}(z, s)$  is considerably simpler. The new coefficient matrix is a linear function of  $z$  while the coefficient matrix for  $\Phi^{(III)}(z, s)$  also has a simple pole at  $z = 0$ ; see (5-28). Indeed, the  $s$ -equation for  $\tilde{\Phi}^{(III)}(z, s)$  is a combination of both equations in the Lax pair (5-27)–(5-28) for  $\Phi^{(III)}(z, s)$  so that the pole at  $z = 0$  cancels out and we have

$$(5-55) \quad \frac{\partial \tilde{\Phi}^{(III)}}{\partial s} = (zs\sigma_3 + \tilde{B}_0) \tilde{\Phi}^{(III)}(z, s) \equiv \tilde{B}(z) \tilde{\Phi}^{(III)}(z, s),$$

where

$$(5-56) \quad \tilde{B}_0 = 2s^{-k\sigma_3} B_0 s^{k\sigma_3} = \frac{1}{s} \begin{pmatrix} 0 & 2us^{-2k} \\ 2vs^{2k} & 0 \end{pmatrix}.$$

One also can notice that the expansion (5-24) of the function  $\Phi^{(III)}(z, s)$  at  $z = \infty$  transforms to the following expansion of the function  $\tilde{\Phi}^{(III)}(z, s)$  at  $z = \infty$ :

$$(5-57) \quad \tilde{\Phi}^{(III)}(z) = \left( I + \frac{\tilde{\phi}_1^{(III)}}{z} + \dots \right) z^{k\sigma_3} e^{\frac{s^2 z}{2} \sigma_3}, \quad z \rightarrow \infty,$$

where

$$(5-58) \quad \tilde{\phi}_1^{(III)}(s) = \frac{1}{s} s^{-k\sigma_3} \phi_1^{(III)}(s) s^{k\sigma_3} = \phi_1^Z(s^2) + \frac{1}{2} \sigma_3.$$

The function  $\tilde{\Phi}^{(III)}(z, s)$  satisfies a differential equation with respect to  $z$  as well, however we will not need it. What we will need is the difference equation for  $\tilde{\Phi}^{(III)}(z, s)$  associated with the shift  $k \rightarrow k+1$ . To derive this equation we indicate explicitly the dependence of  $\tilde{\Phi}^{(III)}(z, s)$  on  $k$ ,

$$(5-59) \quad \tilde{\Phi}^{(III)}(z, s) \equiv \tilde{\Phi}_k^{(III)}(z, s),$$

and consider the discrete logarithmic derivative  $\tilde{\Phi}_{k+1}^{(III)} [\tilde{\Phi}_k^{(III)}]^{-1}(z)$ . Since the jump matrix of the  $\Phi^{(III)}$ -Riemann-Hilbert problem does not depend on  $k$  we, using again Liouville's theorem, will arrive at the following difference equation:

$$(5-60) \quad \tilde{\Phi}_{k+1}^{(III)}(z) = (zU_0 + U_1^{(k)}) \tilde{\Phi}_k^{(III)}(z) \equiv U_k(z) \tilde{\Phi}_k^{(III)}(z).$$

The matrix coefficients  $U_0$  and  $U_1$  can be determined via the substitution of the expansion (5-57) into (5-60). One finds

$$(5-61) \quad U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$(5-62) \quad U_1^{(k)} = \tilde{\phi}_1^{(III, k+1)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{\phi}_1^{(III, k)} \equiv \begin{pmatrix} a_{k+1} - a_k & -b_k \\ c_{k+1} & 0 \end{pmatrix},$$

where  $\tilde{\phi}_1^{(III, k)}$  is the matrix coefficient from the expansion (5-57) with the explicit indication of the dependence on the integer  $k$ , and  $a_k, b_k, c_k$  are temporary notations for the matrix entries of  $\tilde{\phi}_1^{(III)}$ , i.e.,

$$(5-63) \quad a(s) := (\tilde{\phi}_1^{(III)}(s))_{11} = (\phi_1^Z(s^2))_{11} + \frac{1}{2},$$

$$(5-64) \quad b(s) := (\tilde{\phi}_1^{(III)}(s))_{12} = (\phi_1^Z(s^2))_{12} = -s^{-2-2k}u(s),$$

and

$$(5-65) \quad c(s) := (\tilde{\phi}_1^{(III)}(s))_{21} = (\phi_1^Z(s^2))_{21} = -s^{-2+2k}v(s).$$

In presenting these formulae we have also taken into account equations (5-32) and (5-58)

The next step is to consider the compatibility condition of equations (5-60) and (5-55), that is, the differential-difference equation

$$(5-66) \quad \frac{dU_k(s)}{ds} = \tilde{B}_{k+1}(z)U_k(z) - U_k(z)\tilde{B}_k(z).$$

This equation, in particular, means that

$$(5-67) \quad -\frac{db_k}{ds} = 2sb_k(a_{k+1} - a_k).$$

From formulae (5-52), (5-49), and (5-64) it follows that

$$(5-68) \quad h_k(s) = -2\pi i(\phi_1^{YIII}(s))_{12} = -2\pi i s^{1+2k}(\phi_1^Z(s^2))_{12} = -2\pi i s^{1+2k}b_k(s),$$

and hence (5-67) becomes

$$(5-69) \quad \frac{d}{ds} \log h_k = \frac{1+2k}{s} - 2s(a_{k+1} - a_k).$$

By virtue of the first equation in (5-52), we immediately derive from (5-69) the differential identity

$$(5-70) \quad \frac{d}{ds} \log H_k^{(III)}(s) = -2sa_k + \frac{k^2 - 1}{s} + c(s)$$

for the Hankel determinant  $H_k^{(III)}$ , where

$$(5-71) \quad c(s) = \frac{d}{ds} \log H_1^{(III)} + 2sa_1.$$

Because of (5-63) and (5-49), to complete the proof of the Lemma we only need to show that

$$(5-72) \quad c(s) = s + \frac{1}{s}.$$

In order to see (5-72) we notice first that

$$(5-73) \quad H_1^{(III)}(s) = 2\pi i I_1(2s).$$

Secondly, we use the fact that

$$(5-74) \quad a_1 = s^{-1}c_0 + \frac{1}{2},$$

where  $c_0$  is the zero-degree coefficient in the orthogonal polynomial

$$(5-75) \quad P_1(z) = z + c_0, \quad \int_{\Gamma_1} P_1(z)w_0^{(III)}(z)dz = 0,$$

and hence

$$(5-76) \quad c_0 = -\frac{I_2(2s)}{I_1(2s)}.$$

Equation (5-72) follows from (5-73), (5-74), (5-76), and one of the classical differential identities for the Bessel functions,

$$\frac{dI_1(s)}{ds} = \frac{1}{s}I_1(s) + I_2(s).$$

□

Combining (5-49), (5-42), and Lemma 4, we arrive at the following expression of  $\frac{d}{dt} \log H_k^{(III)}(\sqrt{t})$  in terms of the Painlevé III function  $\xi(t)$ :

$$(5-77) \quad \frac{d}{dt} \log H_k^{(III)}(\sqrt{t}) = \frac{k^2}{t} + \frac{1}{2} + \frac{\xi(t)}{t}.$$

A comparison of this relation and the asymptotics (5-43) implies the following transition formula from the Laguerre-Hankel determinant  $H_k$  to the Bessel-Hankel determinant  $H^{(III)}$ :

$$(5-78) \quad \frac{d}{dt} \log H_k \left( \frac{t}{N} \right) \xrightarrow{N \rightarrow \infty} \frac{d}{dt} \log H_k^{(III)}(\sqrt{t}) - \frac{k^2}{t},$$

or, more explicitly,

$$(5-79) \quad \frac{d}{dt} \log \det \left[ L_{N+k-1-(i+j)}^{2k-1} \left( -\frac{t}{N} \right) \right] \xrightarrow{N \rightarrow \infty} \frac{d}{dt} \log \det \left[ I_{i+j+1}(2\sqrt{t}) \right] - \frac{k^2}{t}.$$

**Remark 1.** Formula (5-77), in slightly different but equivalent form, was first obtained by Forrester and Witte in [18]. If one could prove the asymptotic relation (5-79) directly, then our second main result, i.e. Theorem 2, may be obtained via a simple reference to [18]. However, a direct asymptotic analysis of the Laguerre-Hankel determinant is not a simple matter; one can easily see, for instance, that all the matrix entries have the same leading behavior as  $N \rightarrow \infty$ . As it stands at the moment, (5-79) is a non-trivial by-product of our Riemann-Hilbert analysis. We note that [1] employs a different analysis which does lead directly to (5-79) and so does enable the results of [18] to be applied straightforwardly. But this method is not so well-suited to giving exact formulae for finite  $N$ , so should be considered complementary to ours. We also note that the confluence of confluent hypergeometric function solutions of Painlevé V to Bessel function solutions of Painlevé III has been studied by Masuda [36]. It would be interesting to see if the degeneration method of Masuda could provide an alternative proof of (5-79), which does not seem to be an immediate corollary of the constructions of [36].

## 6. CONFORMAL BLOCK EXPANSION OF THE $\tau$ -FUNCTION

The function  $\tau_L$  introduced in [35] is defined as

$$(6-1) \quad x \frac{d}{dx} \log \tau_L := \sigma_L + \frac{\theta_*(x + \theta_*)}{2},$$

where  $\sigma_L(x)$  satisfies the  $\sigma$ -form of the Painlevé V equation

$$(6-2) \quad \left( x \frac{d^2 \sigma_L}{dx^2} \right)^2 = \left( \sigma_L - x \frac{d\sigma_L}{dx} + 2 \left( \frac{d\sigma_L}{dx} \right)^2 \right)^2 - \frac{1}{4} \left( \left( 2 \frac{d\sigma_L}{dx} - \theta_* \right)^2 - 4\tilde{\theta}_0^2 \right) \left( \left( 2 \frac{d\sigma_L}{dx} + \theta_* \right)^2 - 4\theta_t^2 \right),$$

where  $\theta_*$ ,  $\theta_t$ , and  $\tilde{\theta}_0$  are complex parameters. To identify the above equation with the  $\sigma$ -form in (3-21), we need to make the shift

$$(6-3) \quad \sigma = \tilde{\sigma} + x \frac{(2\theta_0 + \theta_\infty)}{4} + \frac{(2\theta_0 + \theta_\infty)^2}{8}$$

so that

$$(6-4) \quad \left( x \frac{d^2 \tilde{\sigma}}{dx^2} \right)^2 = \left[ \tilde{\sigma} - x \frac{d\tilde{\sigma}}{dx} + 2 \left( \frac{d\tilde{\sigma}}{dx} \right)^2 \right]^2 - \frac{1}{4} \left( \left( 2 \frac{d\tilde{\sigma}}{dx} + \frac{\theta_\infty}{2} \right)^2 - \theta_0^2 \right) \left( \left( 2 \frac{d\tilde{\sigma}}{dx} - \frac{\theta_\infty}{2} \right)^2 - \theta_1^2 \right).$$

Comparing (6-2) and (6-4) we have that  $\tilde{\sigma} = \sigma_L$  if

$$(6-5) \quad 2\theta_* = \theta_\infty, \quad 4\theta_t^2 = \theta_0^2, \quad 4\tilde{\theta}_0^2 = \theta_1^2$$

or

$$(6-6) \quad 2\theta_* = -\theta_\infty, \quad 4\theta_t^2 = \theta_1^2, \quad 4\tilde{\theta}_0^2 = \theta_0^2.$$

Next, we consider the relations between the Jimbo-Miwa  $\tau$ -function (3-22) and  $\tau_L$ . By (6-2) and (6-3) we have

$$(6-7) \quad x \frac{d}{dx} \log \tau = x \frac{d}{dx} \log \tau_L - \frac{x}{4} (2\theta_* + \theta_\infty) + \frac{\theta_0^2 + \theta_1^2}{4} - \frac{\theta_\infty^2}{8} - \frac{\theta_*^2}{2}.$$

We choose  $2\theta_* = -\theta_\infty$  so that the relation between the two  $\tau$ -functions becomes

$$(6-8) \quad \tau(x) = c \tau_L(x) x^{\frac{\theta_0^2 + \theta_1^2 - \theta_\infty^2}{4}}$$

for some constant  $c$ . Therefore, the correspondence of the set of parameters  $(\theta_0, \theta_1, \theta_\infty)$  and  $(\theta_*, \theta_t, \tilde{\theta}_0)$  is as in (6-6). There is still an ambiguity in identifying the sign of the parameters  $\theta_t$  and  $\tilde{\theta}_0$ , but this is not important for our purpose because the conformal block expansion is symmetric with respect to  $\theta_t \rightarrow -\theta_t$  and  $\tilde{\theta}_0 \rightarrow -\tilde{\theta}_0$ .

From (3-71), the parameters of the Painlevé equation we are considering are

$$(6-9) \quad \theta_* = -\frac{1}{2}\theta_\infty = k, \quad \theta_t = -\frac{1}{2}\theta_1 = \frac{k+N}{2}, \quad \tilde{\theta}_0 = \frac{1}{2}\theta_0 = \frac{k+N}{2}.$$

Comparing (4-22), (4-23), and (4-24) we have the relation between the functions  $f_k$  and  $\tau_L$

$$(6-10) \quad x \frac{d}{dx} \log f_k = x \frac{d}{dx} \log \tau_L + \frac{(k+N)^2}{2} - k^2 - \frac{k}{2}x.$$

Next, we present the conformal block expansion of the function  $\tau_L$  near  $x = 0$  as developed in [35]. For this purpose we introduce for any positive integer  $N$  the partition

$$(6-11) \quad \lambda := \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N > 0\}.$$

Partitions can be identified in the obvious way with Young diagrams. The set of all Young diagrams will be denoted by  $\mathbb{Y}$ . For  $\lambda \in \mathbb{Y}$ ,  $\lambda'$  denotes the transposed diagram,  $\lambda_i$  and  $\lambda'_j$  the number of boxes in the  $i$ th row and  $j$ th column of  $\lambda$ , and  $|\lambda|$  the total number of boxes. Given a box  $(i, j) \in \lambda$ , its hook length is defined as  $h_\lambda(i, j) := \lambda_i + \lambda'_j - i - j + 1$ , and for the empty partition,  $h_\emptyset(i, j) = 1$ . For complex numbers  $\theta_*$ ,  $\tilde{\theta}_0$ ,  $\theta_t$ , and  $\sigma$  and partitions  $\lambda$  and  $\mu$  let us introduce the quantity

$$(6-12) \quad \begin{aligned} \mathcal{B}_{\lambda, \mu}(\tilde{\theta}_0, \theta_t, \theta_*, \sigma) &:= \prod_{(i, j) \in \lambda} \frac{(\theta_* + \sigma + i - j) \left( (\theta_t + \sigma + i - j)^2 - \tilde{\theta}_0^2 \right)}{h_\lambda^2(i, j) \left( \lambda'_j + \mu_i - i - j + 1 + 2\sigma \right)^2} \\ &\times \prod_{(i, j) \in \mu} \frac{(\theta_* - \sigma + i - j) \left( (\theta_t - \sigma + i - j)^2 - \tilde{\theta}_0^2 \right)}{h_\mu^2(i, j) \left( \lambda_i + \mu'_j - i - j + 1 - 2\sigma \right)^2}. \end{aligned}$$

**Theorem 3** ([35]). *The  $\tau$ -function of the Painlevé V equation has the following expansion near  $x = 0$ :*

$$(6-13) \quad \tau_L(x) = \mathcal{N}_0 \sum_{n \in \mathbb{Z}} e^{2\pi i n \eta} \mathcal{C}_0(\theta_t; \tilde{\theta}_0; \theta_*; \sigma + n) \mathcal{B}(\theta_t; \tilde{\theta}_0; \theta_*; \sigma + n; x),$$

where  $\sigma, \eta$  correspond to the initial conditions,  $\mathcal{N}_0$  is a constant,  $\mathcal{B}(\theta_t; \tilde{\theta}_0; \theta_*; \sigma; x)$  is given by the combinatorial series

$$(6-14) \quad \mathcal{B}(\theta_t; \tilde{\theta}_0; \theta_*; \sigma; x) := x^{\sigma^2 - \tilde{\theta}_0^2 - \theta_t^2} e^{-\theta_t x} \sum_{\lambda, \mu \in \mathbb{Y}} \mathcal{B}_{\lambda, \mu}(\tilde{\theta}_0, \theta_t, \theta_*, \sigma; \sigma) x^{|\lambda| + |\mu|}$$

with  $\mathcal{B}_{\lambda,\mu}(\tilde{\theta}_0, \theta_t, \theta_*, \sigma)$  as in (6-12), and the structure constants  $\mathcal{C}_0(\theta_t; \tilde{\theta}_0; \theta_*; \sigma)$  are expressed in terms of the Barnes  $G$ -function as

$$(6-15) \quad \mathcal{C}_0(\theta_t; \tilde{\theta}_0; \theta_*; \sigma) := \prod_{\epsilon=\pm 1} \frac{G(1 + \theta_* + \epsilon\sigma) G(1 + \tilde{\theta}_0 + \theta_t + \epsilon\sigma) G(1 - \tilde{\theta}_0 + \theta_t + \epsilon\sigma)}{G(1 + 2\epsilon\sigma)},$$

where  $\theta_* = -\frac{1}{2}\theta_\infty$ ,  $\theta_t = -\frac{1}{2}\theta_1$ , and  $\tilde{\theta}_0 = \frac{1}{2}\theta_0$ .

Comparing the expansion near  $x = 0$  of  $\tau_L$  in (6-13) with (6-10) and (4-24), and using the fact that

$$(6-16) \quad \sum_{\lambda, \mu \in \mathbb{Y}} \mathcal{B}_{\lambda, \mu}(\tilde{\theta}_0, \theta_t, \theta_*, \sigma) x^{|\lambda|+|\mu|} = 1 + \left( \theta_t + \frac{1}{2}\theta_* + \frac{\theta_*}{2\sigma^2}(\theta_t^2 - \tilde{\theta}_0^2) \right) x + \mathcal{O}(x^2),$$

we obtain that  $\sigma = k$  and  $\eta = 0$  and, furthermore, the sum in (6-13) is only over non-negative integers  $n$ . Unfortunately, for the values  $\sigma = k$ ,  $\theta_* = k$ ,  $\theta_t = \tilde{\theta}_0 = \frac{k+N}{2}$ , the structure constants  $\mathcal{C}_0(\theta_t; \tilde{\theta}_0; \theta_*; \sigma)$  are undefined. For this reason we need first to take a limit, using the following relation for the Barnes  $G$  function that holds for non-negative integers  $n$ :

$$(6-17) \quad G(1 + \delta - n) = \delta^n (-1)^{\frac{n(n-1)}{2}} G(1 + n) + O(\delta^{n+\epsilon}), \quad n \in \mathbb{N}_0, \epsilon, \delta > 0.$$

Then we have that, for  $\sigma = k + n$  a non-negative integer,

$$(6-18) \quad \begin{aligned} \tilde{\mathcal{C}}_0(\theta_t; k; \sigma) &:= \lim_{\delta \rightarrow 0} 2^{-2k} (-1)^{\frac{k(k+1)}{2}} \delta^k \mathcal{C}_0(\theta_t; \theta_t; k; \sigma - \delta) \\ &= \frac{G(1 + k + \sigma) G(1 + 2\theta_t + \sigma) G(1 + 2\theta_t - \sigma) G(1 + \sigma)^2}{(-1)^{\sigma(\sigma+k)} 2^{2\sigma-2k} G(1 + 2\sigma)^2}, \end{aligned}$$

where we observe that we have the freedom to multiply the structure constants by  $\sigma$ -independent quantities. Furthermore, the  $\tau_L$  function is defined up to the constant  $\mathcal{N}_0$  that we obtained from (4-20), namely, we must have

$$(6-19) \quad F_N(0, k) = \frac{G(N + 2k + 1) G(N + 1) G(k + 1)^2}{G(N + k + 1)^2 G(2k + 1)} = f_k(0) = \mathcal{N}_0 \tilde{\mathcal{C}}_0\left(\frac{N+k}{2}; k; k\right),$$

which implies

$$(6-20) \quad \mathcal{N}_0 = \frac{1}{G(N + k + 1)^2}.$$

We also observe that  $\tilde{\mathcal{C}}_0\left(\frac{N+k}{2}; k; N + k + 1\right) = 0$ , namely, we have only  $N + 1$  conformal blocks. We arrive at the following conjectural expression.

**Conjecture 1.** *The function  $f_k(x)$  defined in (4-3) has the following conformal block expansion near  $x = 0$ :*

$$(6-21) \quad \begin{aligned} f_k(x) &= \frac{e^{-\frac{N}{2}x - kx}}{G(N + k + 1)^2} \sum_{n=0}^N \tilde{\mathcal{C}}_0\left(\frac{N+k}{2}; k; k + n\right) x^{2nk + n^2} \\ &\quad \times \sum_{\lambda, \mu \in \mathbb{Y}} \mathcal{B}_{\lambda, \mu}\left(\frac{k+N}{2}, \frac{k+N}{2}, k, k + n\right) x^{|\lambda|+|\mu|}, \end{aligned}$$

where the coefficients  $\tilde{\mathcal{C}}_0\left(\frac{N+k}{2}; k; k + n\right)$  are defined in (6-18) and the conformal blocks are defined in (6-12).



The coefficients  $\beta_{2j}$  in (4-25) for  $j = 1, \dots, k$  of the power series expansion of  $f_k(x)$  near zero are obtained from the first conformal block  $\mathcal{B}_{\lambda, \mu} \left( \frac{k+N}{2}, \frac{k+N}{2}, k, k \right)$ . This conformal block contains the term  $(i-j)$  in the sum over the partition  $\mu$  and therefore it is nonzero only for the empty partition. Furthermore, the factor  $(k+i-j)$  in the product over boxes of  $\lambda$  reduces the summation in (6-21) to Young diagrams with  $\lambda_1 \leq k$ . Therefore, the sum over the first conformal block reduces to

$$\begin{aligned}
 f_k(x) &= \frac{\tilde{\mathcal{C}}_0 \left( \frac{N+k}{2}; k; k \right)}{G(N+k+1)^2} e^{-\frac{N}{2}x-kx} \sum_{\lambda, \mu \in \mathbb{Y}} \mathcal{B}_{\lambda, \mu} \left( \frac{k+N}{2}, \frac{k+N}{2}, k, k \right) x^{|\lambda|+|\mu|} + \dots \\
 (6-22) \quad &= f_k(0) e^{-\frac{N}{2}x-kx} \sum_{\substack{\lambda \in \mathbb{Y}, \\ \lambda_1 \leq k}} \prod_{(i,j) \in \lambda} \frac{(2k+i-j)(N+2k+i-j)(k+i-j)}{h_\lambda^2(i,j) \left( \lambda'_j - i - j + 1 + 2k \right)^2} x^{|\lambda|} + \dots \\
 &= f_k(0) (1 + b_2 x^2 + b_4 x^4 + b_6 x^6 + b_8 x^8 + b_{10} x^{10} + \dots),
 \end{aligned}$$

where the coefficients  $b_{2j}$  coincides with the coefficients  $\beta_{2j}$  defined in (4-25) and (4-26), namely

$$\begin{aligned}
 b_2 &:= \frac{N(2k+N)}{8-32k^2} = \beta_2, \quad b_4 := \frac{N(2k+N)(2kN+N^2+2)}{128(16k^4-40k^2+9)} = \beta_4, \\
 b_6 &:= -\frac{N(2k+N)}{3072 \left( (1-4k^2)^2 (16k^4-136k^2+225) \right)} \\
 &\quad \times (16k^4 N^2 + 16k^3 N (N^2+3) + 4k^2 (N^4-3N^2+16) \\
 &\quad + 4kN (5-9N^2) - 9N^4 + 10N^2 - 16) = \beta_6, \\
 b_8 &:= \frac{N(2k+N)(2kN+N^2+6)}{98304 (1-4k^2)^2 (64k^6-1328k^4+7564k^2-11025)} \\
 (6-23) \quad &\quad \times (16k^4 N^2 + 16k^3 N (N^2+3) + 4k^2 (N^4-27N^2+40) \\
 &\quad + 4kN (29-33N^2) - 33N^4 + 58N^2 - 40) = \beta_8, \\
 b_{10} &:= -\frac{N(2k+N)}{3932160(4k^2-81)(4k^2-49)(4k^2-25)(4k^2-9)^2(4k^2-1)^2} \\
 &\quad \times (256k^8 N^4 + 512k^7 N^5 + 2560k^7 N^3 + 384k^6 N^6 - 1920k^6 N^4 + 14080k^6 N^2 \\
 &\quad + 128k^5 N^7 - 9600k^5 N^5 - 23040k^5 N^3 + 48640k^5 N + 16k^4 N^8 - 8320k^4 N^6 \\
 &\quad - 28208k^4 N^4 - 67200k^4 N^2 + 86016k^4 - 2880k^3 N^7 + 20064k^3 N^5 \\
 &\quad - 20960k^3 N^3 - 54016k^3 N - 360k^2 N^8 + 31288k^2 N^6 + 82960k^2 N^4 \\
 &\quad + 70768k^2 N^2 - 215040k^2 + 11976kN^7 + 52920kN^5 + 97776kN^3 \\
 &\quad - 149280kN + 1497N^8 + 8820N^6 + 24444N^4 - 74640N^2 + 48384) = \beta_{10}.
 \end{aligned}$$

We remark that the combinatorial expression for the coefficient  $b_{2k} = \beta_{2k}$  provided by the conformal block expansion (6-22) should be consistent with the combinatorial expression obtained in [12].

Finally, we want to consider the limit  $N \rightarrow \infty$  as done in [20] that reduces the  $\tau_L$  function of the Painlevé V equation to the  $\tau$  function of the Painlevé III equation. In our case we have that the quantity

$$(6-24) \quad \frac{f_k\left(\frac{t}{N}\right)}{N^{k^2}}$$

has a well-defined term-by-term limit as  $N \rightarrow \infty$  that can be easily obtained using the properties of the Barnes  $G$ -function (see Appendix C) and the fact that  $\mathcal{B}_{\lambda, \mu} \left( \frac{k+N}{2}, \frac{k+N}{2}, k, k \right) \left( \frac{t}{N} \right)^{|\lambda|+|\mu|}$  has

a well-defined limit that can be calculated term-wise. Therefore, we can define the function

$$(6-25) \quad \tau_{III}(t) := \lim_{N \rightarrow \infty} \frac{f_k(\frac{t}{N})}{N^{k^2}}.$$

The function  $\tau_{III}(t)$  has a conformal block expansion

$$(6-26) \quad \tau_{III}(t) = e^{-\frac{t}{2}} \sum_{n=0}^{\infty} \mathcal{C}_{III}(k; k+n) t^{2nk+n^2} \sum_{\lambda, \mu \in \mathbb{Y}} \mathcal{B}_{\lambda, \mu}^{III}(k, k+n) t^{|\lambda|+|\mu|},$$

where

$$(6-27) \quad \mathcal{C}_{III}(k; \sigma) = \frac{G(1+k+\sigma)G(1+\sigma)^2}{(-1)^{\sigma(\sigma+k)} 2^{2\sigma-2k} G(1+2\sigma)^2},$$

and

$$(6-28) \quad \begin{aligned} \mathcal{B}_{\lambda, \mu}^{III}(\theta_*, \sigma) &= \prod_{(i,j) \in \lambda} \frac{(\theta_* + \sigma + i - j)(\sigma + i - j)}{h_{\lambda}^2(i, j) \left( \lambda'_j + \mu_i - i - j + 1 + 2\sigma \right)^2} \\ &\times \prod_{(i,j) \in \mu} \frac{(\theta_* - \sigma + i - j)(-\sigma + i - j)}{h_{\mu}^2(i, j) \left( \lambda_i + \mu'_j - i - j + 1 - 2\sigma \right)^2}. \end{aligned}$$

The function  $\xi(t) := t \frac{d}{dt} \log \tau_{III}(t)$  satisfies the  $\sigma$ -form of the Painlevé III equation (1-14) with boundary conditions  $\xi(0) = 0$  and  $\xi'(0) = 0$ . We write the first few terms of the expansion of  $\tau_{III}(t)$  near  $t = 0$ :

$$(6-29) \quad \tau_{III}(t) = \frac{G(1+k)^2}{G(1+2k)} \left[ -\frac{t^2}{4(4k^2-1)2!} + \frac{t^4}{16(4k^2-1)(4k^2-9)4!} + \right.$$

$$(6-30) \quad \left. -\frac{15t^6}{64(4k^2-1)^2(4k^2-25)6!} + t^8 \frac{105(4k^2-33)}{256(4k^2-1)^2(4k^2-9)(4k^2-25)(4k^2-49)8!} \right.$$

$$(6-31) \quad \left. -t^{10} \frac{925(16k^4-360k^2+1497)}{1024(4k^2-1)^2(4k^2-9)^2(4k^2-25)(4k^2-49)(4k^2-81)10!} + \mathcal{O}(t^{11}) \right].$$

One can observe that the function  $\tau_{III}(t)$  exactly reproduces the coefficients  $F(h, k)$ . In particular, we observe that it is sufficient to consider only the first conformal block to obtain the coefficients  $F(h, k)$  for  $k > h - \frac{1}{2}$ . The quantity  $\mathcal{B}_{\lambda, \mu}^{III}(k, k)$  vanishes for any non-empty partition  $\mu$ . Therefore, it is sufficient to sum only over Young diagrams  $\lambda$ . Furthermore, the factor  $(k+i-j)$  in the product over boxes of  $\lambda$  reduces the summation in (6-28) to Young diagrams with  $\lambda_1 \leq k$ .

**Conjecture 2.** *We have the following conjectural relation for the function  $F(h, k)$  defined in (1-4):*

$$(6-32) \quad F(h, k) = (-1)^h \frac{G(k+1)^2}{G(2k+1)} (2h)! \sum_{\substack{\lambda \in \mathbb{Y} \\ |\lambda|=2h, \lambda_1 \leq k}} \prod_{(i,j) \in \lambda} \frac{(2k+i-j)(k+i-j)}{h_{\lambda}^2(i, j) \left( \lambda'_j - i - j + 1 + 2k \right)^2},$$

with  $k > h - \frac{1}{2}$ .

In [12] a combinatorial formula for  $F(h, k)$  has been obtained and the first few terms of this formula are compatible with ours.

## APPENDIX A. DIFFERENTIAL-DIFFERENCE IDENTITIES AND THE PROOF OF LEMMA 1

The jump matrix of the  $\Psi$ -Riemann-Hilbert problem (3-44)–(3-48) is not only independent of  $z$  and  $x$ , but also independent of  $n$ . Therefore, if we indicate the dependence of  $\Psi(z)$  on  $n$  as

$$(A.1) \quad \Psi(z) \equiv \Psi_n(z),$$

we can state that the discrete logarithmic derivative  $\Psi_{n+1}\Psi_n^{-1}(z)$  is also analytic in  $\mathbb{C} \setminus (\{0\} \cup \{1\})$ . In fact, since the singular right factors in the right-hand sides of formulae (3-46) and (3-48) are also  $n$ -independent, we conclude that the only singularity of  $\Psi_{n+1}\Psi_n^{-1}(z)$  is the simple pole at  $z = 0$ . Therefore, we conclude that, in addition to the two differential equations (3-53) and (3-54), the function  $\Psi(z, x) \equiv \Psi_n(z, x)$  satisfies the difference equation

$$(A.2) \quad \Psi_{n+1}(z) = \left( \frac{1}{z} U_{-1} + U_0^{(n)} \right) \Psi_n(z) \equiv U_n(z) \Psi_n(z).$$

Let us determine the structure of the matrix coefficients  $U_{-1}$  and  $U_0^{(n)}$  in (A.2). We consider a more detailed form of the expansion (3-47),

$$(A.3) \quad \Psi_n(z) = \begin{pmatrix} (-1)^n & 0 \\ 0 & (-1)^{k+N-n} \end{pmatrix} (I + zM_1 + \cdots) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n-3k-N} \end{pmatrix}, \quad z \rightarrow 0,$$

where

$$(A.4) \quad M_1 \equiv \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} -n + \frac{x}{2} & 0 \\ 0 & n - k - N - \frac{x}{2} \end{pmatrix} - m_1^{(n)}.$$

Here  $m_1^{(n)}$  is the matrix coefficient from the expansion (3-34) with the explicit indication of the dependence on the integer  $n$ . Remembering the relation of the coefficient  $m_1$  with the norm  $h_n$  of the orthogonal polynomials  $P_n(z)$  (see (3-33)), we note that

$$(A.5) \quad \begin{aligned} a_n &= -n + \frac{x}{2} - (m_1^{(n)})_{11}, & b_n &= -(m_1^{(n)})_{12} = \frac{1}{2\pi i} h_n, \\ c_n &= -(m_1^{(n)})_{21} = \frac{2\pi i}{h_{n-1}}, & d_n &= n - k - N - \frac{x}{2} - (m_1^{(n)})_{22}. \end{aligned}$$

Observe that, in particular,

$$(A.6) \quad b_n c_{n+1} = 1.$$

Plugging (A.3) into the right-hand side of the equation  $U_n = \Psi_{n+1} \Psi_n^{-1}$ , we have that

$$(A.7) \quad \begin{aligned} U_n &= \begin{pmatrix} (-1)^{n+1} & 0 \\ 0 & (-1)^{k+N-n-1} \end{pmatrix} (I + zM_1^{(n+1)} + \cdots) \begin{pmatrix} z^{-n-1} & 0 \\ 0 & z^{n+1-3k-N} \end{pmatrix} \\ &\times \begin{pmatrix} z^n & 0 \\ 0 & z^{-n-3k-N} \end{pmatrix} (I - zM_1^{(n)} + \cdots) \begin{pmatrix} (-1)^n & 0 \\ 0 & (-1)^{k+N-n} \end{pmatrix} \\ &= -\frac{1}{z} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_n - a_{n+1} & (-1)^{N+k} b_n \\ -(-1)^{N+k} c_{n+1} & 0 \end{pmatrix} + \mathcal{O}(z), \quad z \rightarrow 0. \end{aligned}$$

Hence,

$$(A.8) \quad U_{-1} = -\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$(A.9) \quad U_0^{(n)} = \begin{pmatrix} a_n - a_{n+1} & (-1)^{N+k} b_n \\ -(-1)^{N+k} c_{n+1} & 0 \end{pmatrix}.$$

Let us rewrite the Lax pair (3-53)–(3-54) as

$$(A.10) \quad \frac{\partial \Psi_n}{\partial z} = \left( x A_\infty^{(n)} + \frac{A_0^{(n)}}{z} + \frac{A_1^{(n)}}{z-1} \right) \Psi_n \equiv A_n(z) \Psi_n(z),$$

$$(A.11) \quad \frac{\partial \Psi_n}{\partial x} = z A_\infty^{(n)} \Psi_n \equiv B_n(z) \Psi_n(z),$$

indicating explicitly the dependence of all objects on  $n$ . The compatibility of the equations (A.10) and (A.11) with the difference equation (A.2) yields the differential-difference zero-curvature equations

$$(A.12) \quad \frac{\partial U_n}{\partial z} = A_{n+1}(z) U_n - U_n A_n(z)$$

and

$$(A.13) \quad \frac{\partial U_n}{\partial x} = B_{n+1}(z) U_n - U_n B_n(z).$$

From (A.12) and (A.13) it follows that

$$(A.14) \quad A_\infty^{(n+1)} U_0^{(n)} - U_0^{(n)} A_\infty^{(n)} = 0$$

and

$$(A.15) \quad \frac{d}{dx} U_0^{(n)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A_\infty^{(n)} - A_\infty^{(n+1)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Defining the matrix  $A_\infty$  as

$$(A.16) \quad A_\infty := \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & -\alpha_n \end{pmatrix}$$

and using (A.9), we have from (A.14) that

$$(A.17) \quad \begin{aligned} (a_n - a_{n+1})(\alpha_{n+1} - \alpha_n) &= (-1)^{N+k}(\beta_{n+1} + \gamma_n b_n), \quad \gamma_{n+1} b_n + c_{n+1} \beta_n = 0, \\ \alpha_{n+1} b_n (-1)^{N+k} - \beta_n (a_n - a_{n+1}) + (-1)^{N+k} b_n \alpha_n &= 0, \\ \gamma_{n+1} (a_n - a_{n+1}) + (-1)^{N+k} \alpha_{n+1} c_{n+1} + (-1)^{N+k} \alpha_n c_{n+1} &= 0. \end{aligned}$$

Also, from (A.15),

$$(A.18) \quad (-1)^{N+k} \frac{d b_n}{dx} = \beta_n$$

or

$$(A.19) \quad \frac{1}{b_n} \frac{d b_n}{dx} = (-1)^{N+k} \beta_n \frac{1}{b_n}.$$

Using (A.6), one also obtains that

$$(A.20) \quad \frac{d}{dx} \log b_n = (-1)^{N+k} c_{n+1} \beta_n.$$

Recall now the relation between the functions  $\Psi_n(z)$  and  $\Phi_n(z)$ ,

$$(A.21) \quad \Psi_n(z) = Y^{(n)}(1) \Phi_n(z) z^{-\frac{3k+N}{2}} (z-1)^{\frac{k+N}{2}}.$$

Taking into account (3-65), we conclude from (A.21) that, as  $z \rightarrow \infty$ ,

$$\begin{aligned}
 \Psi_n(z) &= Y^{(n)}(1) \left( I + \frac{\phi_1^{(n)}}{z} + \dots \right) e^{\frac{xz}{2}\sigma_3} \begin{pmatrix} z^k & 0 \\ 0 & z^{-k} \end{pmatrix} z^{-k} \left( 1 - \frac{N+k}{2} \frac{1}{z} + \dots \right) \\
 (A.22) \quad &= Y^{(n)}(1) \left( I + \frac{1}{z} \left( \phi_1^{(n)} - \frac{N+k}{2} I \right) + \dots \right) e^{\frac{xz}{2}\sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & z^{-2k} \end{pmatrix} \\
 &= Y^{(n)}(1) \left( I + \frac{1}{z} \widehat{\phi}_1^{(n)} + \dots \right) e^{\frac{xz}{2}\sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & z^{-2k} \end{pmatrix},
 \end{aligned}$$

where

$$(A.23) \quad \widehat{\phi}_1^{(n)} := \phi_1^{(n)} - \frac{N+k}{2} I.$$

Let us plug this expansion into the difference equation (A.2):

$$(A.24) \quad Y^{(n+1)}(1) \left( I + \frac{1}{z} \widehat{\phi}_1^{(n+1)} + \dots \right) = \left[ -\frac{1}{z} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + U_0^{(n)} \right] Y^{(n)}(1) \left( I + \frac{1}{z} \widehat{\phi}_1^{(n)} + \dots \right).$$

We then see that

$$(A.25) \quad Y^{(n+1)}(1) = U_n^{(0)} Y^{(n)}(1)$$

and

$$(A.26) \quad Y^{(n+1)}(1) \widehat{\phi}_1^{(n+1)} = U_n^{(0)} Y^{(n)}(1) \widehat{\phi}_1^{(n)} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y^{(n)}(1).$$

Excluding  $Y^{(n+1)}(1)$ , we arrive at the formula

$$(A.27) \quad \widehat{\phi}_1^{(n+1)} - \widehat{\phi}_1^{(n)} \equiv \phi_1^{(n+1)} - \phi_1^{(n)} = - \left[ Y^{(n)}(1) \right]^{-1} \left[ U_n^{(0)} \right]^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y^{(n)}(1).$$

Put

$$(A.28) \quad Y^{(n)}(1) := \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix}.$$

Note that since  $\det Y^{(n)}(1) = 1$  we have

$$(A.29) \quad p_n s_n - q_n r_n = 1.$$

With these notations and recalling (A.9), we have from (A.27) that

$$\begin{aligned}
 (A.30) \quad &\phi_1^{(n+1)} - \phi_1^{(n)} \\
 &= - \begin{pmatrix} s_n & -q_n \\ -r_n & p_n \end{pmatrix} \begin{pmatrix} 0 & -(-1)^{N+k} b_n \\ (-1)^{N+k} c_{n+1} & a_n - a_{n+1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix} \\
 &= - \begin{pmatrix} -q_n (-1)^{N+k} c_{n+1} & -(-1)^{N+k} s_n b_n + q_n (a_{n+1} - a_n) \\ (-1)^{N+k} p_n c_{n+1} & r_n b_n (-1)^{N+k} + p_n (a_n - a_{n+1}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix} \\
 &= - \begin{pmatrix} -(-1)^{N+k} q_n c_{n+1} & 0 \\ (-1)^{N+k} p_n c_{n+1} & 0 \end{pmatrix} \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix}.
 \end{aligned}$$

In particular,

$$(A.31) \quad (\phi_1^{(n+1)})_{11} - (\phi_1^{(n)})_{11} = (-1)^{N+k} p_n q_n c_{n+1}.$$

On the other hand, from (3-56) it follows that

$$(A.32) \quad A_\infty^{(n)} = \frac{1}{2} \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} s_n & -q_n \\ -r_n & p_n \end{pmatrix} = \frac{1}{2} \begin{pmatrix} p_n s_n + q_n r_n & -2p_n q_n \\ -2p_n r_n & -s_n p_n - r_n q_n \end{pmatrix}.$$

Comparing these equations with (A.16), we conclude that

$$(A.33) \quad \beta_n = -p_n q_n.$$

Therefore, (A.31) can be rewritten as

$$(A.34) \quad (\phi_1^{(n+1)})_{11} - (\phi_1^{(n)})_{11} = -(-1)^{N+k} \beta_n c_{n+1}$$

which, together with (A.20), yields the important formula

$$(A.35) \quad \frac{d}{dx} \log b_n = -(\phi_1^{(n+1)})_{11} + (\phi_1^{(n)})_{11}$$

or, remembering (A.5),

$$(A.36) \quad \frac{d}{dx} \log h_n = -(\phi_1^{(n+1)})_{11} + (\phi_1^{(n)})_{11}.$$

With equation (A.36) we are ready to prove Lemma 2. Indeed, taking into account that

$$(A.37) \quad \log H_n - \log H_1 = \log h_{n-1} + \log h_{n-2} + \cdots + \log h_1,$$

we have from (A.36) that

$$(A.38) \quad \frac{d}{dx} \log H_n - \frac{d}{dx} \log H_1 = -(\phi_1^{(n)})_{11} + (\phi_1^{(1)})_{11}.$$

Hence, in order to obtain the statement of Lemma 1 one only has to calculate explicitly  $(\phi_1^{(1)})_{11}$  and show that

$$(A.39) \quad (\phi_1^{(1)})_{11} = -\frac{d}{dx} \log H_1 + \frac{N+k}{2}.$$

First, we notice that if we define  $\kappa^{(n)}$  as the matrix coefficient in the expansion

$$(A.40) \quad Y^{(n)}(s) = Y^{(n)}(1) \left( I + \kappa^{(n)}(s-1) + \cdots \right), \quad s \rightarrow 1,$$

then by a straightforward calculation we arrive at the relation

$$(A.41) \quad \phi_1^{(n)} = -\kappa^{(n)} + \frac{N+k}{2} \sigma_3.$$

Indeed, we have that (dropping the indication of the dependence of  $n$ )

$$(A.42) \quad \begin{aligned} \Phi(z) &= Y^{-1}(1) \Psi(z, x) z^{\frac{3k+N}{2}} (z-1)^{-\frac{N+k}{2}} \\ &= Y^{-1}(1) \chi(z) e^{\frac{xz}{2} \sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & z^{(-3k-N)} (z-1)^{N+k} \end{pmatrix} z^{\frac{3k+N}{2}} (z-1)^{-\frac{N+k}{2}} \\ &= Y^{-1}(1) X(z) e^{\frac{xz}{2} \sigma_3} \begin{pmatrix} z^{\frac{3k+N}{2}} (z-1)^{-\frac{N+k}{2}} & 0 \\ 0 & z^{\frac{(-3k-N)}{2}} (z-1)^{\frac{N+k}{2}} \end{pmatrix} \end{aligned}$$

and hence, as  $z \rightarrow \infty$ ,

$$(A.43) \quad \begin{aligned} \Phi(z) &= Y^{-1}(1) X(z) \left( I + \frac{N+k}{2} \frac{1}{z} \sigma_3 + \cdots \right) z^{k\sigma_3} e^{\frac{xz}{2} \sigma_3} \\ &= Y^{-1}(1) Y \left( \frac{z-1}{z} \right) \left( I + \frac{N+k}{2} \frac{1}{z} \sigma_3 + \cdots \right) z^{k\sigma_3} e^{\frac{xz}{2} \sigma_3} \\ &= Y^{-1}(1) Y(1) \left( I + \kappa \left( \frac{z-1}{z} - 1 \right) + \frac{N+k}{2} \frac{1}{z} \sigma_3 + \cdots \right) z^{k\sigma_3} e^{\frac{xz}{2} \sigma_3} \\ &= \left[ I - \frac{1}{z} \left( \kappa - \frac{N+k}{2} \sigma_3 + \cdots \right) \right] z^{k\sigma_3} e^{\frac{xz}{2} \sigma_3}, \end{aligned}$$

where the last equation, in view of the definition (3-17) of  $\phi_1^{(n)}$ , is equivalent to (A.41).

Secondly, from the definition (3-29) of the function  $Y^{(n)}(z)$  we have that

$$(A.44) \quad Y^{(1)}(s) = \begin{pmatrix} P_1(s) & \frac{1}{2\pi i} \int_C \frac{P_1(s')w_0(s')}{s'-s} ds' \\ -\frac{2\pi i}{h_0} P_0(s) & -\frac{1}{h_0} \int_C \frac{w_0(s')}{s'-s} ds' \end{pmatrix},$$

where

$$(A.45) \quad P_0(s) = 1, \quad P_1(s) = s + c, \quad c = -\frac{1}{\int_C w_0(s) ds} \int_C s w_0(s) ds = -\frac{1}{h_0} \int_C s w_0(s) ds.$$

Hence,

$$(A.46) \quad Y^{(1)}(1) = \begin{pmatrix} 1+c & \frac{1}{2\pi i} \int_C \frac{(s+c)w_0(s)}{s-1} ds \\ -\frac{2\pi i}{h_0} & -\frac{1}{h_0} \int_C \frac{w_0(s)}{s-1} ds \end{pmatrix}, \quad Y^{(1)'}(1) = \begin{pmatrix} 1 & \frac{1}{2\pi i} \int_C \frac{(s+c)w_0(s)}{(s-1)^2} ds \\ 0 & -\frac{1}{h_0} \int_C \frac{w_0(s)}{(s-1)^2} ds \end{pmatrix}.$$

For the matrix coefficient  $\kappa$  we have (recall that  $\det Y^{(n)} \equiv 1$ )

$$(A.47) \quad \kappa^{(1)} = Y^{(1)-1}(1)Y^{(1)'}(1) = \begin{pmatrix} -\frac{1}{h_0} \int_C \frac{w_0(s)}{s-1} ds & -\frac{1}{2\pi i} \int_C \frac{(s+c)w_0(s)}{(s-1)^2} ds \\ 2\pi i & 1+c \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2\pi i} \int_C \frac{(s+c)w_0(s)}{(s-1)^2} ds \\ 0 & -\int_C \frac{w_0(s)}{(s-1)^2} ds \end{pmatrix}.$$

Therefore,

$$(A.48) \quad (\kappa^{(1)})_{11} = -\frac{1}{h_0} \int_C \frac{w_0(s)}{s-1} ds$$

and (cf. (A.41))

$$(A.49) \quad (\phi_1^{(1)})_{11} = \frac{N+k}{2} + \frac{1}{h_0} \int_C \frac{w_0(s)}{s-1} ds = \frac{N+k}{2} - \frac{1}{h_0} \int_C \frac{e^{\frac{x}{1-s}}}{(1-s)^{2k+1} s^{N+k}} ds,$$

where

$$(A.50) \quad h_0 = \int_C w_0(s) ds = \int_C \frac{e^{\frac{x}{1-s}}}{(1-s)^{2k} s^{N+k}} ds.$$

Now note that

$$(A.51) \quad H_1 = h_0$$

and

$$(A.52) \quad \frac{d}{dx} \log H_1 = \frac{1}{H_1} \int_C \frac{e^{\frac{x}{1-s}}}{(1-s)^{2k+1} s^{N+k}} ds = \frac{1}{h_0} \int_C \frac{e^{\frac{x}{1-s}}}{(1-s)^{2k+1} s^{N+k}} ds.$$

Equation (A.39) follows from (A.49) and (A.52). This completes the proof of Lemma 1.

## APPENDIX B. A SECOND DERIVATION OF THE SECOND TERM IN THE EXPANSION OF $\sigma_k(x)$

In this appendix we derive the second term of the expansion (3-84) directly from the Laguerre determinant. We write the Laguerre determinant as

$$(B.1) \quad \mathcal{L}(x) := \det \left[ L_{N+k-1-(i+j)}^{(2k-1)}(-x) \right]_{i,j=0,\dots,k-1}.$$

Combining this with (1-9) and (3-82) gives

$$(B.2) \quad \sigma_k(x) = -Nk + \frac{\mathcal{L}'(x)}{\mathcal{L}(x)} x.$$

Therefore the desired term is

$$(B.3) \quad \sigma'_k(0) = \frac{\mathcal{L}'(0)}{\mathcal{L}(0)}.$$



Fix the determinant size  $k$ . Note from (1-5) that each entry of the determinant in (B.1) has the structure

$$(B.4) \quad L_{N+k-1-(i+j)}^{(2k-1)}(-x) = \ell_{i,j} + \frac{N+k-1-(i+j)}{2k} \ell_{i,j} x + \mathcal{O}(x^2), \quad x \rightarrow 0.$$

Here  $\ell_{i,j}$  (which depends on  $N$  and  $k$ ) can be read off from (1-5), although we will not need its particular form. Now  $\mathcal{L}(0)$  is simply the matrix with  $ij$  entry  $\ell_{i,j}$ , and

$$(B.5) \quad \mathcal{L}'(0) = \frac{1}{2k} \begin{vmatrix} (N+k-1)\ell_{0,0} & \ell_{0,1} & \cdots & \ell_{0,k-1} \\ (N+k-2)\ell_{1,0} & \ell_{1,1} & \cdots & \ell_{1,k-1} \\ \vdots & \vdots & & \vdots \\ (N+0)\ell_{k-1,0} & \ell_{k-1,1} & \cdots & \ell_{k-1,k-1} \end{vmatrix} + \frac{1}{2k} \begin{vmatrix} \ell_{0,0} & (N+k-2)\ell_{0,1} & \cdots & \ell_{0,k-2} \\ \ell_{1,0} & (N+k-3)\ell_{1,1} & \cdots & \ell_{1,k-1} \\ \vdots & \vdots & & \vdots \\ \ell_{k-1,0} & (N-1)\ell_{k-1,1} & \cdots & \ell_{k-1,k-1} \end{vmatrix} \\ + \cdots + \frac{1}{2k} \begin{vmatrix} \ell_{0,0} & \ell_{0,1} & \cdots & (N+0)\ell_{0,k-1} \\ \ell_{1,0} & \ell_{1,1} & \cdots & (N-1)\ell_{1,k-1} \\ \vdots & \vdots & & \vdots \\ \ell_{2,0} & \ell_{2,1} & \cdots & (N-k+1)\ell_{k-1,k-1} \end{vmatrix}.$$

A series of straightforward determinant manipulations now shows that  $\mathcal{L}'(0) = \frac{N}{2}\mathcal{L}(0)$ , which proves the proposition. First, notice that using multilinearity we can write each determinant in (B.5) as the sum of  $N\mathcal{L}(0)/2k$  and an  $N$ -independent term. As there are  $k$  determinants, the  $N$ -dependent terms sum to  $N\mathcal{L}(0)/2$ , the expected answer. We now show the  $N$ -independent terms cancel, using  $k=3$  and  $k=2$  to illustrate the odd and even cases, respectively.

If  $k$  is odd, we use multilinearity to shift all the coefficients of  $\ell_{i,j}$  to be the same as the center summand, canceling extra terms in pairs (we drop the common factor of  $1/2k$ ):

$$(B.6) \quad \begin{vmatrix} 2\ell_{0,0} & \ell_{0,1} & \ell_{0,2} \\ 1\ell_{1,0} & \ell_{1,1} & \ell_{1,2} \\ 0\ell_{2,0} & \ell_{2,1} & \ell_{2,2} \end{vmatrix} + \begin{vmatrix} \ell_{0,0} & 1\ell_{0,1} & \ell_{0,2} \\ \ell_{1,0} & 0\ell_{1,1} & \ell_{1,2} \\ \ell_{2,0} & -1\ell_{2,1} & \ell_{2,2} \end{vmatrix} + \begin{vmatrix} \ell_{0,0} & \ell_{0,1} & 0\ell_{0,2} \\ \ell_{1,0} & \ell_{1,1} & -1\ell_{1,2} \\ \ell_{2,0} & \ell_{2,1} & -2\ell_{2,2} \end{vmatrix} \\ = \begin{vmatrix} (1+1)\ell_{0,0} & \ell_{0,1} & \ell_{0,2} \\ (0+1)\ell_{1,0} & \ell_{1,1} & \ell_{1,2} \\ (-1+1)\ell_{2,0} & \ell_{2,1} & \ell_{2,2} \end{vmatrix} + \begin{vmatrix} \ell_{0,0} & 1\ell_{0,1} & \ell_{0,2} \\ \ell_{1,0} & 0\ell_{1,1} & \ell_{1,2} \\ \ell_{2,0} & -1\ell_{2,1} & \ell_{2,2} \end{vmatrix} + \begin{vmatrix} \ell_{0,0} & \ell_{0,1} & (1-1)\ell_{0,2} \\ \ell_{1,0} & \ell_{1,1} & (0-1)\ell_{1,2} \\ \ell_{2,0} & \ell_{2,1} & (-1-1)\ell_{2,2} \end{vmatrix} \\ = \begin{vmatrix} 1\ell_{0,0} & \ell_{0,1} & \ell_{0,2} \\ 0\ell_{1,0} & \ell_{1,1} & \ell_{1,2} \\ -1\ell_{2,0} & \ell_{2,1} & \ell_{2,2} \end{vmatrix} + \begin{vmatrix} \ell_{0,0} & 1\ell_{0,1} & \ell_{0,2} \\ \ell_{1,0} & 0\ell_{1,1} & \ell_{1,2} \\ \ell_{2,0} & -1\ell_{2,1} & \ell_{2,2} \end{vmatrix} + \begin{vmatrix} \ell_{0,0} & \ell_{0,1} & 1\ell_{0,2} \\ \ell_{1,0} & \ell_{1,1} & 0\ell_{1,2} \\ \ell_{2,0} & \ell_{2,1} & -1\ell_{2,2} \end{vmatrix}.$$

Each row now has an associated coefficient  $j$ ,  $j \in \{-(k-1)/2, \dots, (k-1)/2\}$ . These coefficients sum to zero, so we can rewrite  $\mathcal{L}(0)$  by multiplying each row by  $x^j$ :

$$(B.7) \quad \begin{vmatrix} \ell_{0,0} & \ell_{0,1} & \ell_{0,2} \\ \ell_{1,0} & \ell_{1,1} & \ell_{1,2} \\ \ell_{2,0} & \ell_{2,1} & \ell_{2,2} \end{vmatrix} = \begin{vmatrix} x^1\ell_{0,0} & x^1\ell_{0,1} & x^1\ell_{0,2} \\ x^0\ell_{1,0} & x^0\ell_{1,1} & x^0\ell_{1,2} \\ x^{-1}\ell_{2,0} & x^{-1}\ell_{2,1} & x^{-1}\ell_{2,2} \end{vmatrix}.$$

Taking the derivative of both sides with respect to  $x$  and then setting  $x=1$  shows the  $N$ -independent terms (those in (B.6)) are zero.

If  $k$  is even, then there is no center summand. We now shift the coefficients so the coefficient of the top row is  $k/2$  and that of the bottom row is  $-k/2+1$ . Note that one summand already has these coefficients, so while most shifts will cancel in pairs, the final summand introduces a new term  $-(k/2)\mathcal{L}(0)$ :

$$(B.8) \quad \begin{vmatrix} 1\ell_{0,0} & \ell_{0,1} \\ 0\ell_{1,0} & \ell_{1,1} \end{vmatrix} + \begin{vmatrix} \ell_{0,0} & 0\ell_{0,1} \\ \ell_{1,0} & -1\ell_{1,1} \end{vmatrix} = \begin{vmatrix} 1\ell_{0,0} & \ell_{0,1} \\ 0\ell_{1,0} & \ell_{1,1} \end{vmatrix} + \begin{vmatrix} \ell_{0,0} & 1\ell_{0,1} \\ \ell_{1,0} & 0\ell_{1,1} \end{vmatrix} - 1 \begin{vmatrix} \ell_{0,0} & \ell_{0,1} \\ \ell_{1,0} & \ell_{1,1} \end{vmatrix}.$$

We can now rewrite  $x^{k/2}\mathcal{L}(0)$  as  $\mathcal{L}(0)$  with each row multiplied by  $x^j$ , where  $j$  is the appropriate coefficient:

$$(B.9) \quad x^1 \begin{vmatrix} \ell_{0,0} & \ell_{0,1} \\ \ell_{1,0} & \ell_{1,1} \end{vmatrix} = \begin{vmatrix} x^1 \ell_{0,0} & x^1 \ell_{0,1} \\ x^0 \ell_{1,0} & x^0 \ell_{1,1} \end{vmatrix}.$$

Differentiating both sides with respect to  $x$  and setting  $x = 1$  shows the  $N$ -independent terms are zero if  $k$  is even.

### APPENDIX C. THE BARNES $G$ -FUNCTION

The Barnes  $G$ -function is defined as the infinite product

$$(C.1) \quad G(1+z) := (2\pi)^{\frac{z}{2}} \exp\left(-\frac{z+z^2(1+\gamma)}{2}\right) \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right)^k \exp\left(\frac{z^2}{2k}-z\right),$$

where  $\gamma$  is Euler's constant. The Barnes  $G$ -function satisfies the functional equation

$$(C.2) \quad G(1+z) = \Gamma(z) G(z).$$

It is analytic in the whole complex plane and has the following asymptotic expansion as  $|z| \rightarrow \infty$ ,  $\arg z \neq \pi$ :

$$(C.3) \quad \log G(1+z) = \left(\frac{z^2}{2} - \frac{1}{12}\right) \log z - \frac{3z^2}{4} + \frac{z}{2} \log 2\pi + \zeta'(-1) + \mathcal{O}(z^{-2}).$$

It satisfies the useful relation

$$(C.4) \quad \frac{G(1+z+n)G(1-z)}{G(1-z-n)G(1+z)} = (-1)^{\frac{n(n+1)}{2}} \left(\frac{\pi}{\sin \pi z}\right)^n, \quad n \in \mathbb{Z}.$$

### REFERENCES

- [1] E. Bailey, S. Bettin, G. Blower, J.B. Conrey, A. Prokhorov, M.O. Rubinstein and N.C. Snaith, Mixed moments of characteristic polynomials of random unitary matrices, arXiv:1901.07479.
- [2] P. Bleher and A.R. Its, Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model, *Ann. Math.* **150**, 185–266 (1999).
- [3] F. Bornemann, P.J. Forrester, and A. Mays, Finite size effects for spacing distributions in random matrix theory: circular ensembles and Riemann zeros, *Stud. Appl. Math.* **138**, 401–437 (2017).
- [4] D. Bump and A. Gamburd, On the averages of characteristic polynomials from the classical groups, *Comm. Math. Phys.* **265**, 227–274 (2006).
- [5] P. Clarkson, Special polynomials associated with rational solutions of the fifth Painlevé equation, *J. Comput. Appl. Math.* **178**, 111–129 (2005).
- [6] P. Clarkson, Recurrence coefficients for discrete orthonormal polynomials and the Painlevé equations, *J. Phys. A* **46**, 185205 (2013).
- [7] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein, and N.C. Snaith, Autocorrelation of random matrix polynomials, *Comm. Math. Phys.* **237**, 365–395 (2003).
- [8] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein, and N.C. Snaith, Integral moments of L-functions, *Proc. Lond. Math. Soc. (3)* **91**, 33–104 (2005).
- [9] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein, and N.C. Snaith, Lower order terms in the full moment conjecture for the Riemann zeta function, *J. Number Theory* **128**, 1516–1554 (2008).
- [10] J.B. Conrey, M.O. Rubinstein, and N.C. Snaith, Moments of the derivative of characteristic polynomials with an application to the Riemann zeta function, *Comm. Math. Phys.* **267**, 611–629 (2006).
- [11] J. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, 1990.
- [12] P.-O. Dehayé, Joint moments of derivatives of characteristic polynomials, *Algebra Number Theory* **2**, 31–68 (2008).
- [13] P.-O. Dehayé, A note on moments of derivatives of characteristic polynomials, 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010), 681–692, *Discrete Math. Theor. Comput. Sci. Proc.*, AN, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2010.
- [14] P.A. Deift and X. Zhou, Asymptotics for the Painlevé II equation, *Comm. Pure Appl. Math.* **48**, 277–337 (1995).
- [15] F.J. Dyson, Statistical theory of the energy levels of complex systems I, II, III, *J. Math. Phys.* **3**, 140–175 (1962).

- [16] A.S. Fokas, A.R. Its, A.A. Kapaev, and V.Y. Novokshenov, *Painlevé Transcendents. The Riemann-Hilbert Approach*. Mathematical Surveys and Monographs **128**. American Mathematical Society, Providence, RI, 2006.
- [17] P.J. Forrester and N.S. Witte, Application of the  $\tau$ -function theory of Painlevé equations to random matrices:  $P_V$ ,  $P_{III}$ , the LUE, JUE, and CUE, *Comm. Pure Appl. Math.* **55**, 679–727 (2002).
- [18] P.J. Forrester and N.S. Witte, Boundary conditions associated with the Painlevé III' and V evaluations of some random matrix averages, *J. Phys. A* **39**, 8983–8995 (2006).
- [19] P.J. Forrester and N.S. Witte, Painlevé II in random matrix theory and related fields, *Constr. Approx.* **41**, 589–613 (2015).
- [20] O. Gamayun, N. Iorgov, and O. Lisovyy, How instanton combinatorics solves Painlevé VI, V and IIIs, *J. Phys. A* **46**, 335203 (2013).
- [21] C.P. Hughes, On the characteristic polynomial of a random unitary matrix and the Riemann zeta function, Ph.D. Thesis, University of Bristol (2001).
- [22] C.P. Hughes, Random matrix theory and discrete moments of the Riemann zeta function, *J. Phys. A* **36**, 2907 (2003).
- [23] C.P. Hughes, J.P. Keating, and N. O'Connell, Random matrix theory and the derivative of the Riemann zeta function, *Proc. R. Soc. Lond. A* **456**, 2611–2627 (2000).
- [24] C.P. Hughes, J.P. Keating, and N. O'Connell, On the characteristic polynomial of a random unitary matrix, *Comm. Math. Phys.* **220**, 429–451 (2001).
- [25] A.R. Its, Large  $N$  asymptotics in random matrices, in *Random Matrices, Random Processes and Integrable Systems*, CRM Series in Mathematical Physics, ed. John Harnad, Springer, 2011.
- [26] A.R. Its, Painlevé transcendents, in *The Oxford Handbook of Random Matrix Theory*, eds. G. Akemann, J. Baik, and P. Di Francesco, Oxford University Press, 2011.
- [27] M. Jimbo and T. Miwa, Monodromy preserving deformations of linear ordinary differential equations with rational coefficients. II, *Physica D* **2**, 407–448 (1981).
- [28] M. Jimbo, T. Miwa, and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and  $\tau$ -function, *Physica D* **2**, 306–352 (1981).
- [29] K. Kajiwara and T. Masuda, On the Umemura polynomials for the Painlevé III equation, *Phys. Lett. A* **260**, 462–467 (1999).
- [30] J.P. Keating, Random matrices and number theory, in *Proceedings of the Les Houches Summer School on Applications of Random Matrices in Physics, Les Houches 2004*, eds. E. Brezin et al., Springer, 2005.
- [31] J.P. Keating, Random matrices and number theory: some recent themes, in *Stochastic Processes and Random Matrices: Lecture Notes of the Les Houches Summer School: Volume 104, July 2015*, eds. G. Schehr, A. Altland, Y.V. Fyodorov, N. O'Connell, and L.F. Cugliandolo, 2017.
- [32] J.P. Keating and N.C. Snaith, Random matrix theory and  $\zeta(1/2 + it)$ , *Comm. Math. Phys.* **214**, 57–85 (2000).
- [33] J.P. Keating and N.C. Snaith, Random matrix theory and  $L$ -functions at  $s = 1/2$ , *Comm. Math. Phys.* **214**, 91–110 (2000).
- [34] J.P. Keating and N.C. Snaith, Random matrix theory and number theory, in *The Handbook on Random Matrix Theory*, eds. G. Akemann, J. Baik, and P. Di Francesco, Oxford University Press, 2011.
- [35] O. Lisovyy, H. Nagoya, and J. Roussillon, Irregular conformal blocks and connection formulae for Painlevé V functions, *J. Math. Phys.* **59**, 091409 (2018).
- [36] T. Masuda, Classical transcendental solutions of the Painlevé equations and their degeneration, *Tohoku Math. J.* **56**, 467–490 (2004).
- [37] T. Masuda, Y. Ohta, and K. Kajiwara, A determinant formula for a class of rational solutions of Painlevé V equation, *Nagoya Math. J.* **168**, 1–25 (2002).
- [38] H.L. Montgomery, The pair correlation of zeros of the zeta function, *Proc. Symp. Pure Math.* **24**, 181–193 (1973).
- [39] M. Noumi and Y. Yamada, Umemura polynomials for the Painlevé V equation, *Phys. Lett. A* **247**, 65–69 (1998).
- [40] K. Okamoto, Polynomial Hamiltonians associated with Painlevé equations. I, *Proc. Japan Acad. Ser. A Math. Sci.* **56**, 264–268 (1980).
- [41] K. Okamoto, Polynomial Hamiltonians associated with Painlevé equations. II, *Proc. Japan Acad. Ser. A Math. Sci.* **56**, 367–371 (1980).
- [42] K. Okamoto, Studies on the Painlevé equations. II. Fifth Painlevé equation  $P_V$ , *Japan. J. Math.* **13**, 47–76 (1987).
- [43] K. Okamoto, Studies on the Painlevé equations IV. Third Painlevé equation  $P_{III}$ , *Funkcial. Ekvac.* **30**, 305–332 (1987).
- [44] H. Riedtmann, A combinatorial approach to mixed ratios of characteristic polynomials, arXiv:1805.07261 (2018).
- [45] N.C. Snaith, Riemann zeros and random matrix theory, *Milan J. Math.* **78**, 135–152 (2010).
- [46] B. Winn, Derivative moments for characteristic polynomials from the CUE, *Comm. Math. Phys.* **315**, 531–562 (2012).

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